Explicitly Simple Near-tie Auctions

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Abstract. We consider the problem of truthfully auctioning a single item, that can be either fractionally or probabilistically divided among several winners when their bids are sufficiently close to a tie.

While Myerson's Lemma states that any monotone allocation rule can be implemented, truthful payments are computed by integrating over each profile, and may be difficult to comprehend and explain. We look for payment rules that are given explicitly as a simple function of the allocated fraction and the others' bids. For two agents, this simply coincides with (non-negative) Myerson's payments. For three agents or more, we characterize the near-tie allocation rules that admit such explicit payments, and provide an iterative algorithm to compute them. In particular we show that any such payment rule must require positive payments to some of the bidders.

Keywords: Mechanism design · Diversity · Simplicity

1 Introduction

Consider a single-item auction with several participants, and suppose the leading bids are \$100 and \$94. While the first two bids are almost tied, the outcome for the bidders is very different: The first one gets the item (and pays for it), whereas the second one gets nothing. Awarding the item to the second highest bidder with some probability (say 0.25), or dividing it $3/4$ and $1/4$ (when possible) would not have a serious impact on the social welfare, but it would greatly increase the satisfaction of the runner-up. However, when gaps are larger (say, bids are \$100 and \$43), we may prefer to allocate everything to a single winner. Two questions arise from this example:

1. What would be reasonable allocation rules that partially award the non-highest bidder(s)?
2. Given such allocation rules, can we implement those rules with truthful dominant strategies?

A particular case of interest, that includes the above example, is a class of allocation rules we call *near-tie*. Near-tie allocation rules differentiate between two scenarios: (i) a 'Default' case, where the item is allocated to the highest bidder, and (ii) a 'near-Tie' case that is triggered when bids are sufficiently close, thus diversifying the allocation between the highest bidders.

Clearly, increasing the diversity in the set of winning bidders comes at a cost of both welfare and revenue (as we already know that truthful welfare-optimal auctions will only allocate to the highest bidder). However, in many theoretical and practical cases there are additional properties that need to be evaluated and traded-off with welfare and revenue. For example, awarding some fraction of the item to high-enough bidders would (beyond diversification amongst the set of winners) disincentivize fraudulent behavior (e.g., bid leakage [6, 10]), increase egalitarian welfare, and increase bidder retention. Some additional examples from the literature include item utilization [13]; Egalitarian welfare [26]; No-Envy [20, 21]; and bidder dropout [11, 16].

So far, we have discussed how we can design allocation rules that can allocate the item to *high-enough* bidders and why they are important, but, going back to our second question, can we implement them in truthful dominant strategies?

The classic "Myerson Lemma" [17] already gives a positive answer to this question. Indeed, Myerson fully characterizes the set of dominant strategy incentive-compatible allocation and payment rules: any monotone allocation rule can be implemented (including 'near-tie' rules); and implementation is unique under the requirements of individual rationality and no positive transfers.

Unfortunately, the payment rule characterized by the Myerson Lemma is quite complex and can be problematic for implementation in practical scenarios [15, 18]. Moreover, complex mechanisms can directly affect the ability of bidders to best-respond and, in turn, impact their truthful implementation [12, 19].

In this work, we suggest describing the payment rule explicitly using linear functions of the bids, inspired by the common use of *model trees* to represent functions in AI [7, 22]. Informally, in an *explicitly simple* payment rule, the decision tree is only required to decide the allocation, which itself completely determines the payment as a single linear combination of bids. In contrast, a complex rule would require a long list of cases (a larger subtree) with a distinct linear payment function in each case. We believe this description is a natural extension of linear payment functions, which are common in the auction literature.

Using the explicit complexity framework, we focus first on the basic Myerson payment rule (i.e. without positive transfers), showing that it is only explicitly simple in very restricted cases. We then turn to study and characterize other payment rules that are explicitly simple while maintaining our requirement of strategyproofness and individual-rationality.

1.1 Contribution

- We show that basic Myerson's payments are explicitly simple if, and only if, the item is divided amongst the two leading bidders.
- We characterize the conditions under which a near-tie auction for n agents has an explicitly simple implementation.
- We prove that when the item is divided amongst 3 or more agents, any explicit implementation (except in a very specific case we identify) *must require* positive transfers to agents.

The second result shows that it is possible to implement any partition of the item where a near-tie *reduces* the portion of the leading agent and *benefits* all others. We further provide an iterative algorithm that calculates the explicit formula for the payment of each agent as a linear function of the other bids. For all other cases, we show there is no explicit implementation of near-tie auctions (of the kind we study).

1.2 Related Literature

Welfare, revenue, and Myerson. Auctions are typically designed to maximize social welfare and/or revenue. In the case of a single item allocation, welfare maximization is straight forward—the item should be given to the agent with the highest value, and this can be implemented by a second price auction which allocates the item to the highest bidder. A general formula for optimizing revenue (given agents' value distributions) was developed by Myerson, who suggested to transform agents' valuations into 'virtual values' and then optimal strategyproof auction is maximizing these virtual values [17]. In the case of a single-item auction, this always means selecting the single agent with the maximal virtual value, provided that it is not negative. The simplest example of an optimal auction is a second price auction with a reserve price for i.i.d. agents. Note that the revenue-maximizing auction may not be efficient in terms of welfare since, given the reserve price, it may decide to not allocate the auctioned item. The tradeoff between social welfare and revenue has naturally attracted much attention in the literature, see e.g. [5].

Beyond Welfare and Revenue. There may be other considerations on top of welfare and revenue that guide the allocation and/or the choice of the payment rule. Some examples from the literature include item utilization [13]; Egalitarian welfare [26]; redistribution [2, 8]; Equal rights lower bound [9]; No-Envy [20, 21]; and bidder dropout [11, 16].

These issues may be of importance on their own, but also indirectly affect welfare and revenue in the long term, by decreasing competition among bidders (in the case of bidders dropping out) or by intervention of regulation authorities.

Crucially, some of these models are special cases of Myerson's, meaning the desired class of rules must be contained in the set Myerson characterized. Yet, the additional required properties mean that Myerson's lemma, on its own, does not provide the answer (although it may play a part in the solution). Our paper joins this line of work. We further note that some of these papers present negative results, highlighting conditions under which an auction cannot be both truthful and obtain the desired criterion.

Diverse allocations. There are different motivations in literature that justify the study of mechanisms that allocate the items to non-highest bidders, either through fractional or probabilistic allocation rules. The bottom line of this literature is that giving to the non-highest bidders a positive probability of winning the auction will increase participation and satisfaction within the bidders, resulting in better long-term welfare and revenue results. One such example is starvation/bidder dropout. Lee et al. [11] observe that:

> "Applying traditional auction mechanisms ... may result in an inevitable starvation for resources for certain customers ... customers may decide to drop out of the future auction rounds, thereby decreasing the long-term demand."

In their paper, they consider auctions in a sequential setting with a particular dropout model, and suggest an optimal solution. In contrast, we propose to diversify the set of winners as a heuristic solution that is also likely to mitigate the problem of bidder dropout.

Another interesting scenario is represented by the *contests* literature. In a contest, partitioning the item/prize is often desirable as this may incentivize low-chance participants to invest more effort. We note that in contrast to auctions, the probabilistic allocation of a prize in contests is not a design decision, but rather an assumption on the relation between effort and outcome in a given environment. A designer may however decide on the size of the prize and/or decide to partition it up front into several prizes. The common wisdom in contests (Clark and Riis [3]) is that:

> ‘‘an income maximizing contest administrator obtains the most rent seeking contributions when he makes available a single, large prize”

However, this is not always true. Eden [4] assumes the designer can decide on a smaller shared prize in case of a tie between two or more contestants, and artificially increase the chance of a tie (e.g. by deciding that sufficiently close grades are tied). It turns out that under some conditions, such a ‘noisy tie’ can induce more effort from contestants, in equilibrium.

Representation and simplicity. Various authors have argued that mechanisms should be simple to understand or interact with, with different interpretations of what this simplicity means. E.g., Nuñez [19] writes:

> “The idea of transparency is based on the cognitive ability of each player to understand the consequences of his actions. The more transparent a mechanism, the less cognitively complex for an agent to compute his best responses.”

One important argument Nuñez cites for transparency, is that agents may differ in their ability to understand the mechanism, where some cannot even figure out their best responses. Indeed, in the auction literature, ‘simple mechanisms’ are often such with a succinct description, e.g. by few posted prices [23, 24].

Milgrom [14] considers a different notion of simplicity by comparing the strategy spaces of the participants and the resulting equilibria. A mechanism A is simpler than B if its strategy space is contained and it does not introduce new equilibria.

Our notion of explicit mechanisms is motivated more by the arguments of Nũnez that refer to the mechanism itself, than by the ones by Milgrom which consider the simplicity of the strategy space, as in our case the strategy is just reporting the private value. The representation we suggest is inspired by *model trees* [22], which have ample uses in AI and some in economics (see e.g. []).

2 Preliminaries

We consider a setting with a set of n bidders $N = \{1, \dots, n\}$ that compete for a single item. Each bidder $i \in N$ has a non-negative value denoted by $v_i \in \mathbb{R}_{\geq 0}$. We will assume that bidders are sorted in non-increasing order of their values, e.g., $v_1 \geq v_2 \geq \dots \geq v_n$. We refer to i as the *position* or *rank* of the bidder with value v_i.

A *bid profile* $b \in \mathbb{R}^n_{\geq 0}$ is a vector of n non-negative bids, e.g., $b = (b_1, \dots, b_n)$. Except when explicitly stated otherwise (and then different notation is used), we assume that bids are sorted in a non-increasing order, and all of our definitions are anonymous. I.e. they depend only on the values of bids and their relative position, and not on bidders' identity. We denote by $\mathcal{B}^n$ the set of all (sorted) bid profiles of length n.

Allocation. An allocation function is a function $x : \mathcal{B}^n \to \mathbb{R}^n_{\geq 0}$ mapping bid profiles to allocations. When allocating a single item we have the constraint that $\sum_{j \in N} x_j(b) \leq 1$.

Denote $\Delta(n) := \{p \in \mathbb{R}^n_{\geq 0} : \sum_{i \leq n} p_i = 1\}$. Unless explicitly stated otherwise, we restrict attention to allocation functions that are:

Pareto Efficient if $\forall b \in \mathcal{B}^n, \sum_{j \in N} x_j(b) = 1$, thus the range of x is $\Delta(n)$;
Monotone $b'_j \geq b_j$ means $x_j(b'_j, b_{-j}) \geq x_j(b_j, b_{-j})$;
Homogeneous of degree 0 $\forall c \in \mathbb{R}_{\geq 0}$, and any profile b, $x(c \cdot b) = x(b)$.

Note that anonymity is also implied as the profile contains no information on agents' identity, and we can make sure there are no ties (see Footnote 2).

In order to conveniently capture the dependency between bids and allocations as in the opening example, we divide the allocation function into two stages: first, a function $\tilde{s}$ mapping the bid profile into one of finitely many states S; then, at each state $s \in S$, there is a fixed allocation $p^{(s)} \in \Delta(n)$.

Definition 1. *A* Partitioned Single-item Allocation *(PSA) x is described by a tuple $(S, \tilde{s}, \bar{p})$ such that:*

- $\tilde{s} : \mathcal{B}^n \to S$;
- $\bar{p} = (p^{(s)})_{s \in S}$*, where each* $p^{(s)} \in \Delta(n)$;
- $x(b) = p^{(\tilde{s}(b))}$ *for any profile* b.

Thus $p_i^{(s)}$ is the fraction (or probability) of the item that the i'th highest bidder gets at state s. The set of distributions $\bar{p} = (p^{(s)})_{s \in S}$ is called a *partial PSA*.

Two-state PSAs. In this work we focus mainly on *two-state* PSAs, that is when the set S is partitioned in only two subsets, e.g., $S = \{D, T\}$ (alluding to **D**efault and **T**ies). In the following, we define relevant sub-classes of two-state PSAs:

- A two-state PSA is *separated* if there is $k \in \{1, \ldots, n-1\}$ s.t. $p_j^{(D)} > p_j^{(T)}$ if and only if $j \leq k$ (i.e., the top ranked agents prefer D and the others prefer T);
- A separated PSA is *single-top* if $k = 1$. Otherwise it is *multi-top*;
- A single-top PSA is *near-tie* if $p_1^{(D)} = 1$.

We provide two examples of near-tie PSAs.

Example I. In the example from the introduction we have two states $S = \{D, T\}$, with $p^{(D)} = (1, 0, 0, \ldots)$ and $p^{(T)} = (0.75, 0.25, 0, \ldots)$. The 'near-tie' state T is declared whenever the second bid exceeds half of the highest bid, i.e., $\tilde{s}(b) = T$ iff $b_2 > 0.5 b_1$.

Example II. This will be our running example, with $n = 3$. The winner gets the full item if her bid exceeds twice the sum of the two other bids. Otherwise, the leader gets half the item and the two other agents equally split the remaining half. Formally, we have two states $S = \{D, T\}$, with $p^{(T)} = (0.5, 0.25, 0.25)$ and $p^{(D)} = (1, 0, 0)$. All profiles where $b_1 > 2b_2 + 2b_3$ are mapped to $s = D$ and all other profiles are mapped to $s = T$.

We denote by $\mathcal{B}(s)$ the set of all profiles b s.t. $\tilde{s}(b) = s$, thus $\{\mathcal{B}(s)\}_{s \in S}$ is a partition of $\mathcal{B}^n$. Informally, we say that two states s', s'' are *adjacent* if the sets $\mathcal{B}(s'), \mathcal{B}(s'')$ "touch" one another.

Definition 2. *A* payment rule *is a set of functions* $\mathrm{PAY} = (\mathrm{PAY}_j)_{j\leq n}$*, where* $\mathrm{PAY}_j(b) \in \mathbb{R}_{\geq 0}$ *is the price* j *pays* for the full item. *An* auction *is a pair* (x, PAY).

As with x and p, the index $j \leq n$ refers to the *position* of the agent, not to their identity. That is, $\mathrm{PAY}_1(b)$ is the payment assigned to the highest bidder.

The utility of j with value v_j under auction (x, PAY) with bids b is $x_j(v_j, b) := p_j(b)[v_j - \mathrm{PAY}_j(b)]$. Under truthful bidding we can assume $v_j = b_j$.

Manipulations. Suppose that agent j bids $b'_j \neq v_j$. This may alter her position to some $j' \neq j$. Thus $b' = (b_{-j}, b'_j)$ may induce a different ranking over agents. We use $u'_j(v_j, b')$ for the counterfactual utility of the agent whose real rank is j but may not be ranked j in the reported profile b'. Formally, consider a profile b_{-j} and a (possibly untruthful) bid b'_j. Denote by j' the new rank of b'_j in profile $b' = (b_{-j}, b'_j)$. Then

$$u'_j(v_j, b') := u_{j'}(v_j, b') = x_{j'}(b')[v_j - \mathrm{PAY}_{j'}(b')],$$

that is, the utility that the agent ranked at position j' in profile b' would get, if their value had been v_j. A manipulation for j at profile b is a bid b'_j s.t. $u'_j(v_j, b') > u_j(v_j, b)$.

Local Manipulations. In general, an agent may report a bid $b_j \neq v_j$ such that the state changes, or her rank increases or decreases (by one step or more), or both. However if the bid change is sufficiently small, then the changes are 'local': either a single step up or down in rank; *or* a change to an adjacent state. A formal definition of local moves and manipulations is in the full version. An auction is:

- *Individually rational* (IR) if $u_j(v_j, b) \geq 0$ for all $j \leq n, b \in \mathcal{B}^n$;
- *[Locally] Strategyproof* ([L]SP) if there are no [local] manipulations;
- *No Positive Transfers* (NPT) if $\mathrm{PAY}_j(b) \geq 0$ for all $j \leq n, b \in \mathcal{B}^n$;

We say that an auction *implements [in NPT]* PSA x if it holds IR [,NPT] and SP.

2.1 The Myerson Lemma

According to the Myerson lemma [17], for any monotone PSA x there is a unique auction that is SP, IR and NPT. For a discrete allocation rule, the *Total Myerson Payment* is derived by integrating over all increments in allocation as the bid is increasing.

To be consistent with Myerson's definitions, we use $\underline{k}$ for the *name* of an agent, rather than her position.

Formally, we denote by $x^-_{\underline{k}}(b) := \lim_{\varepsilon\to 0} x_{\underline{k}}(b_{-\underline{k}}, b_{\underline{k}} - \varepsilon)$ the left limit $x_{\underline{k}}$ at b, as a function of $b_{\underline{k}}$.[1] Similarly, $x^+_{\underline{k}}(b)$ is the right limit. Thus $x^-_{\underline{k}}(b), x^+_{\underline{k}}(b)$ differ whenever the allocation to $\underline{k}$ changes. Let:

$$\mathrm{TMP}_{\underline{k}}(b_{\underline{k}}, b_{-\underline{k}}) := \sum_{\ell=1}^{L(b_{\underline{k}})} y_\ell[x^+_{\underline{k}}(y_\ell, b_{-\underline{k}}) - x^-_{\underline{k}}(y_\ell, b_{-\underline{k}})], \tag{1}$$

where $y_1, \ldots, y_{L(b_{\underline{k}})}$ are all points in the range $[0, b_{\underline{k}}]$ where the allocation $x_{\underline{k}}$ changes.

[1] For a PSA, $x_{\underline{k}}$ is a step function. In the general case Myerson's lemma uses an integral over $x_{\underline{k}}$ rather than a sum.

$$TMP_{\underline{c}}(b) = \begin{cases} 0 & \text{if } b_{\underline{c}} < 0.5 \\ \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} & \text{if } b_{\underline{c}} \in (0.5, 3) \\ \frac{1}{8} + 3 \cdot \frac{1}{4} = \frac{7}{8} & \text{if } b_{\underline{c}} \in (3, 8) \\ \frac{7}{8} + 8 \cdot \frac{1}{2} = 4\frac{7}{8} & \text{if } b_{\underline{c}} > 8 \end{cases}$$

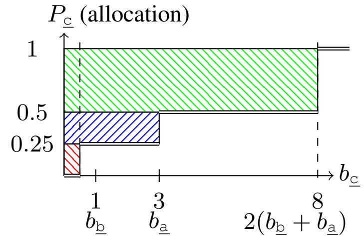


Fig. 1: The total Myerson payment in Example II, under specific bids. The cutoff points are all points where allocation changes: $y_3 = 3$ is a tie profiles (in $y_2 = 1$ there is no change); and $y_1 = 0.5, y_4 = 8$ are the boundary profiles, where the state changes from D to T (when $\underline{c}$ is ranked third) and then from T to D (when $\underline{c}$ is ranked first). The contribution of each cutoff point to the TMP appears in the same color in the formula on the left and in the diagram on the right.

Lemma 1 (Myerson's Lemma [17]). *Let x be a PSA.*

1. *x can be implemented if and only if it is monotone;*
2. *A payment rule implements x if and only if the payment of each agent $\underline{k}$ can be written as $p^0_{\underline{k}}(b_{-\underline{k}}) + \text{TMP}_{\underline{k}}(b)$ for some function $p^0_{\underline{k}} : \mathcal{B}^{n-1} \to \mathbb{R}$;*
3. *The unique payment rule implementing x in NPT is $\text{TMP} = (\text{TMP}_{\underline{k}})_{\underline{k} \in N}$.*

Example. Suppose we fix in our Example II two of the bids at $(b_{\underline{a}} = 3, b_{\underline{b}} = 1)$. The payment rule $\text{TMP}_{\underline{c}}$ for the third agent bidding $b_{\underline{c}}$ is then computed in Fig. 1. E.g. under bidding profile $b = (3, 1, 6)$ agent c will be ranked first, get $x_{\underline{c}}(b) = x_1(b) = \frac{1}{2}$ of the item and will pay $\frac{7}{8}$. Note that in our notation this means the per-unit price for the leading bidder is $\text{PAY}_1(b) = \frac{3}{4}$.

The reason we can always translate $\text{TMP}_{\underline{k}}$ (which is based on agent's index) to PAY_j (which is based on agent's position) is that the Myerson's auction is anonymous. We only need to switch the indices according to the rank of $\underline{k}$, and divide by the allocation (since the total payment of j is $x_j \text{PAY}_j$).

We refer to all payment rules of the form $(p^0_{\underline{k}}(b_{-\underline{k}}) + \text{TMP}_{\underline{k}}(b))_{\underline{k} \in N}$ as the *Myerson class*, to the payment rule PAY induced from TMP (i.e. with $p^0 \equiv 0$) as *Myerson's payments*, and to (x, PAY) as *Myerson's auction*.

2.2 Explicit representation

A payment function PAY_j is *linear* if it is a linear combination of $b_{-j} = (b_i)_{i \neq j}$.

Some common auction rules specify in a clear way how much the winner (or winners) will pay, as a function of the other bids. The immediate example is of course second-price auction, but this true more generally e.g. for GSP and VCG ad-auctions, where every winner pays the next bid, or a fixed linear combination of all lower bids [25].

Ideally, we would like to have a similar explicit description for the payments rules we derive for PSAs. However, while the Myerson lemma enables us to compute the payment for every profile, it is not clear if it has such an explicit formula that can be presented up front: e.g. the solution in Fig. 1 requires multiple cases, and this is for just a single profile $b_{-\underline{c}} = (1,3)$. For other bids, we will get a different conditional description of the payment. Such a payment rule is not particularly convenient to work with, or to explain to naïve participants.

In contrast, we can think of an *explicit representation* of a payment rule as a decision tree for every agent position: the first level of the tree decides the allocation (state). In a simple payment rule, this level would be enough to specify a linear payment rule. A more complicated rule will require a larger subtree at every state, with a single linear payment function in each leaf.

Definition 3. *An* explicit representation *of a payment function* PAY_j *is a tuple* $(Z_j, \tilde{z}_j, \bar{P}_j)$ *such that:*

- $\tilde{z}_j : \mathcal{B}^n \to Z$*;*
- $\bar{\mathrm{PAY}}_j = (\mathrm{PAY}_j^{(z)})_{z \in Z}$ *where each* $\mathrm{PAY}_j^{(z)}$ *is linear;*
- $\mathrm{PAY}_j(b) = \mathrm{PAY}_j^{(\tilde{z}(b))}(b_{-j})$ *for any profile b.*

W.l.o.g each Z_j is a refinement of S, since except in degenerate cases, each state will require a different payment. Hence we refer to S and Z as *states* and *substates*, respectively. We denote by $Z_j(s) \subseteq Z_j$ the set of substates that compose the state s (namely, all payment formulae that may be used under a certain allocation).

Tree/Tabular form. The tree representation of PAY_j sets $\tilde{s}(b)$ in the first level, and $\tilde{z}_j(b)$ in the second level (all substates of $z \in Z_j(s)$ are direct children of s). Then at every leaf $z \in Z_j$ there is the linear payment rule:

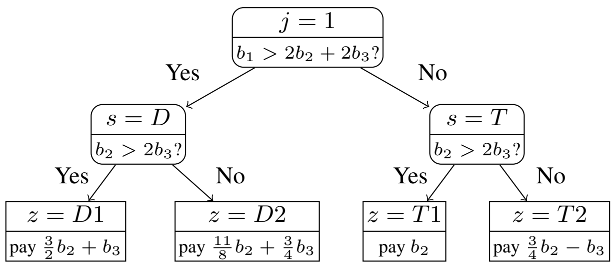
$$\mathrm{PAY}_j^{(z)}(b_{-j}) = \sum_{i<j} t_j^{(z)[i]} b_i + \sum_{i>j} t_j^{(z)[i-1]} b_i. \tag{2}$$

Intuitively, $t_j^{(z)[i]} \in \mathbb{R}$ is the weight that agent j assigns to the i'th highest bid in b_{-j} in substate $z \in Z_j$. Note that we use a subscript j for the rank, and a superscript (z) for the substate. We do not assume weights are positive or normalized.

Therefore every payment rule is described by $n \times \max_j |Z_j| \times (n-1)$ real numbers, which we can conveniety put in a table.

An explicit representation of Myerson's payment rule. It is not hard to see that Myerson's payments can always be written explicitly as a finite tabular form: the number of cutoff points must be finite, since by monotonicity, a bidder at rank j can pass through each substate at most once. Moreover, at every entry of the table, $\mathrm{TMP}_{\underline{k}}(b)$ is a sum of terms of the form $x_{\underline{k}}(\cdot) y_\ell$, where $x_{\underline{k}}(\cdot) = p_j^{(z)}$ for the appropriate rank and substate (i.e. a constant), and y_ℓ is a linear combination of bids, as in Fig. 1. Thus for all profiles b where $\tilde{z}_j(b) = z$, $\mathrm{TMP}_j(b_j, b_{-j})$ is a weighted combination of b_{-j}, where $w_j^{(z)} \in \mathbb{R}^{n-1}$ is some fixed weight vector. We can then extract our explicit representation as $t_j^{(z)} := w_j^{(z)} / p_j^{(z)}$ (as the total payment holds $\mathrm{TMP}_j(b) = p_j^{(z_j(b))} \cdot \mathrm{PAY}_j(b)$).

The explicit complexity of Myerson's payments for Example II is 2 since some states are further divided into two substates. See Fig. 2 . Intuitively, this is since the



j	$t_j^{(D1)}$	$t_j^{(D2)}$	$t_j^{(T1)}$	$t_j^{(T2)}$
1	$\frac{3}{2}, 1$	$\frac{11}{8}, \frac{3}{4}$	$1, 0$	$\frac{3}{4}, -1$
2	$0, 0$		$0, 0$	$\frac{1}{2}, -1$
3	$0, 0$		$0, 0$	$\frac{1}{2}, -1$

Fig. 2: Myerson's payments for Example II in tree form (left, position 1 only) and in tabular form (right). Note that the substates for different positions j need not be the same.

red rectangle in Fig. 1 exists only on some profiles, and thus we get different linear combinations (different substates of s).

Given a PSA $x = (S, \tilde{s}, \bar{p})$, we define the *explicit complexity* of a payment rule that implements x as $\max_{j \leq n, s \in S} |Z_j(S)|$, namely the size of the largest subtree. If the explicit complexity is 1, we say that PAY is *explicitly simple*.

By the Myerson lemma, if there exists such an explicitly simple payment rule, it must belong in the Myerson class, i.e. there are some functions $(p^0_{\underline{k}})_{\underline{k} \in N}$ that modify each agents' payment. However it is not clear a-priori under what conditions such functions exist, how to find them, and what would be the resulting (simple) tabular form. Characterizing these conditions will be our primary goal in this work.

Single state PSAs. We argue that the problem of complexity only arises in non-trivial PSAs, i.e. when there is more than one state.

Consider a single-state auction (i.e. where each agent gets a fixed fraction of the item according to her rank). According to Eq. (1), the agent ranked k'th goes through exactly $L(b_k) = n - k$ cutoff points—one for every agent she passes on the way up. This induces a fixed payment function which is a weighted combination of all lower bids—namely the good old VCG payment.

As we already saw above, this does no longer hold even for two states.

3 Complexity of the Myerson Payment Rule

In this section we show that Myerson's rule is explicitly simple for two agents (regardless of the allocation rule), but becomes substantially more complex as the number of agents grows.

3.1 Two Agents

Theorem 1. *Every 2-agent PSA has a unique explicitly simple NPT implementation, which is the Myerson payment rule.*

Proof sketch. Myerson's lemma states the unique payment rule that is SP, IR and NPT. Thus all that is left is to provide an explicitly simple representation to Myerson payment rule as $(t_1^{(s)}, t_2^{(s)})_{s=1,\dots,c}$.

We order states such that $s = 1$ is the state closest to a tie, and $s = c$ applies when the gap is widest. Let $\beta^{(s)} := \frac{b_2^{(s)}}{b_1^{(s)}}$ at the boundary profile $b^{(s)}$ between states $s, s+1$. We calculate the weights, starting with the lower agent 2.

At any profile $b \in \mathcal{B}(s)$ where $b_1 > b_2$, we have $\text{TMP}_{\underline{j}}(b) = p_j^{(s)}\text{PAY}_j(b) = p_j^{(s)} t_j^{(s)} b_{-j}$. We then apply Eq. (1) and get

$$t_2^{(s)} = \frac{1}{b_1 p_2^{(s)}} \text{TMP}_{\underline{\text{b}}}(b) = \frac{1}{b_1 p_2^{(s)}} \sum_{\ell=1}^{L(b_{\underline{\text{b}}})} y_\ell [x_{\underline{\text{b}}}^+(y_\ell, b_{\underline{\text{a}}}) - x_{\underline{\text{b}}}^-(y_\ell, b_{\underline{\text{a}}})]$$
$$= \frac{1}{b_1 p_2^{(s)}} \sum_{s'=c-1}^{s} (\beta^{(s')} b_1)[p_2^{(s')} - p_2^{(s'+1)}]$$

where the second transition is since the allocation increment points y_ℓ are exactly the boundary profiles where $b_{\underline{\text{b}}} = \beta^{(s')} b_{\underline{\text{a}}}$. Thus,

$$t_2^{(s)} = \sum_{s'=c-1}^{s} \beta^{(s')} \frac{p_2^{(s')} - p_2^{(s'+1)}}{p_2^{(s)}} = \frac{p_2^{(s+1)}}{p_2^{(s)}} t_2^{(s+1)} + \beta^{(s)} (1 - \frac{p_2^{(s+1)}}{p_2^{(s)}}).$$

We then continue to compute weights for agent 1 from the tie state $s = 1$ to $s = c$, getting $t_1^{(1)} = \frac{p_2^{(1)}}{p_1(1)} t_2^{(1)} + (1 - \frac{p_2^{(1)}}{p_1(1)})$ and:

$$t_1^{(s)} = \frac{p_1^{(s-1)}}{p_1^{(s)}} t_1^{(s-1)} + \frac{1}{\beta^{(s)}} (1 - \frac{p_1^{(s-1)}}{p_1^{(s)}}).$$ □

3.2 Beyond Two Agents

Proposition 1. *Consider an explicit representation of the Myerson payment rule. For a single-top PSA with n agents, $|Z_j(s)| \leq n$, and this is tight even for a near-tie PSA.*

Intuitively, we show that even in near-tie PSAs, which are the simplest non-trivial allocation rules, the Myerson payment uses a linear combination of the bids whose coefficients depend on how many other bids are above/below the state cutoff point. Since there are n different options, this induces n substates with distinct linear combinations.

Proof (tightness). Consider any generic near-tie PSA (i.e. $p_1^{(T)} > p_2^{(T)} > \dots > p_n^{(T)}$). State T is selected iff $b_1 > 2\sum_{i>1} b_i$. Denote $p_i := p_i^{(T)}$ for short.

To calculate the TMP of the leading agent, we fix the bids of agents $2, \dots, n$ and consider all cutoff points $y \in \mathbb{R}$. These are exactly $Y = \{b_i\}_{i>1} \cup \{b^*, b^{**}\}$, where

$b^* := \frac{1}{2}b_2 - \sum_{i>2} b_i$ is the transition point from D to T, and $b^{**} = 2\sum_{i>1} b_i$ is the transition point from T to D. Note that for any $j \geq 2$, it is easy to find a profile (in fact many profiles) s.t. b^* is strictly between b_j and b_{j+1} (where $b_{n+1} = 0$). It is also easy to find profiles where $b^* < 0$, meaning that the state is T for any bid $b' < b_2$. Thus for any $j = 2, \ldots, n+1$ we denote these profiles as 'type j' profiles.

By summing over the cutoff points Y using Eq. (1), we get that at every type j profile b, the payment of agent 1 is:

- $\text{PAY}_1(b) = \frac{1}{p_1}\text{TMP}_1(b) = \sum_{i=2}^{j} \frac{p_{i-1}-p_i}{p_1} b_2 + \frac{p_j}{p_1} b^*$, if $b \in \mathcal{B}(T)$;
- $\text{PAY}_1(b) = \text{TMP}_1(b) = \sum_{i=2}^{j} (p_{i-1} - p_i) b_2 + p_j b^* + (1 - p_1) b^{**}$, if $b \in \mathcal{B}(D)$.

since this yields a distinct formula for every j, each of the states T, D must split into n substates, so $|Z_1(T)| = |Z_1(D)| = n$, as required. □

The reason that in Fig. 2 the sets $Z_1(T), Z_1(D)$ include only two substates each rather than 3, is that Example II is not generic, as $p_2 = p_3$.

4 Explicitly Simple Auctions

In the remainder of the paper we will focus on explicitly simple payment rules. We therefore only use the set of states S, as $Z_i = S$ for all i. In this section we derive some general properties that will be used for characterization in the later sections.

Exact Ties and Boundary Points. The definitions above look over the cases where $b_j = b_{j'}$ for some j, j'. More generally, the partition of $\mathcal{B}^n$ into $|S|$ can create complications with closed and open sets. E.g. in our running example, the profile $(b_1 = 8, b_2 = 3, b_3 = 1)$ can be treated as belonging to either state.

We will not be assuming anything on the state at the boundary profiles. As for ties, we assume the profile b implicitly contains ranking to use in case of a tie. Since this ranking is affected by the bids, an agent may 'choose' her rank in case of a tie.[2]

Therefore, in the remainder of the paper it will not matter how boundary profiles are classified in the PSA x or how ties are broken. Since the payment function is defined directly on S the utilities in any profile $b \in \mathcal{B}^n$ are well defined.

4.1 Conditions for Strategyproofness

Proposition 2. *An explicit auction is strategyproof if and only if it is locally strategyproof.*

[2] This can be formally implemented, for example, if each agent j reports (in addition to b_j) a rational number r_j. In case of a tie, we rank the tied agents according to $R_j := r_j \cdot \sqrt{\psi_j}$ where ψ_j is the j'th prime. Note that $R_j, R_{j'}$ are never tied (since $\sqrt{\psi_j}$ are linearly independent over the rationals [1]), unbounded on both sides, and that for any $r_{j'} < r_{j''}$ and j there is r_j s.t. $R_{j'} < R_j < R_{j''}$.

The proof is by breaking the difference between the real value and the reported bid into local steps (into an adjacent state or one rank up or down). Then prove by induction on the number of local steps.

In this section we show that to characterize strategyproofness, we essentially only need to show that all agents are indifferent on certain boundary profiles: between two adjacent states, or between two adjacent positions (i.e. in a tie).

The formal definitions and proofs are available in the full version.

An agent that is right on the boundary and can decide the state in a given profile is called *pivotal*. She *connects* the two states s', s'' if whenever she can change s' to s'', she can also bid right on the boundary.

Lemma 2 (Informal). *If j connects s', s'', then the following statements are equivalent:*

- *j does not have a local state manipulation for s', s'';*
- *j is indifferent between s', s'' at any boundary profile.*

Lemma 3 (Informal). *The following statements are equivalent for agent j and state s:*

- *j does not have a local position manipulation to $j+1$;*
- *j is indifferent about her rank in case of a tie with $j+1$.*

One can check that in our Example II, the agents are indifferent at any boundary point. For example at the boundary profile $b = (8, 3, 1)$ we have $u_1^{(T)} = 0.5(8 - 0.75 \cdot 3 + 0.5 \cdot 1) = 3.125$; and $u_1^{(D)} = 8 - 1.375 \cdot 3 - 0.75 \cdot 1 = 3.125$.

5 Two-state PSAs and Near-Ties

In a two-state PSA there are only two states $S = \{T, D\}$ and thus two possible outcome distributions $p^{(T)}$ and $p^{(D)}$. Due to homogeneity, the function $\tilde{s}$ must be a linear function. That is, there are constants $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ s.t. $\tilde{s}(b) = T$ if $\sum_{i \leq n} \alpha_i b_i > 0$ and $\tilde{s}(b) = D$ if $\sum_{i \leq n} \alpha_i b_i < 0$, with some tie-breaking rule. A two-state PSA is thus fully described by a triplet $(p^{(D)}, p^{(T)}, \bar{\alpha})$.

Note that multiplying $\bar{\alpha}$ by a positive constant does not change the rule.

A two-state PSA $(p^{(D)}, p^{(T)}, \bar{\alpha})$ induces two types of agents: type D for which $p_j^{(D)} > p_j^{(T)}$ and type T for which it is the opposite. We denote by $N_D, N_T \subseteq N$ the two subsets of agents. Note that $\alpha_j < 0$ for $j \in N_D$ and $\alpha_j > 0$ for $j \in N_T$.[3] E.g. the cutoff rule in our running example can be defined by $\alpha_1 = -1, \alpha_2 = \alpha_3 = 2$.

Since multiplying $\bar{\alpha}$ by a positive constant does not matter, we assume w.l.o.g. that agent 1 is type D.

[3] We do not allow agents for which $\alpha_j = 0$, as they would change the allocation without being affected.

$$\begin{array}{ccccccccc}
t_n^{(D)} & \leftrightarrow & t_{n-1}^{(D)} & \cdots & t_j^{(D)} & \cdots & t_2^{(D)} & \overset{?}{\longleftrightarrow} & t_1^{(D)} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
t_n^{(T)} & \leftrightarrow & t_{n-1}^{(T)} & \cdots & t_j^{(T)} & \cdots & t_2^{(T)} & \leftrightarrow & t_1^{(T)}
\end{array}$$

Fig. 3: Every $t_j^{(s)}$ is a vector of $n-1$ weights. Every vertical arrow corresponds to an agent that connects the two states. Every horizontal arrow corresponds to adjacent agents in the respective state.

Separated PSAs. Recall that a two-state PSA is *separated* if $j < j'$ for all $j \in N_D, j' \in N_T$ (if all T are above all D we flip the names of the states). A non-separated PSA is *mixed*. In a separated PSA the type T agents get a larger fraction of the item (at the expense of D agents) as the gap between high and low bids becomes smaller. In the remainder of this section we only consider separated PSAs unless stated otherwise.

If only the highest bidder increases her share of the item in state D then we say this is a **single-top** PSA. Any other separated PSA is **multi-top**. In a single-top PSA, only α_1 is negative and thus we denote $\gamma := -\alpha_1 > 0$. A special case of single-top PSA is a **near-tie** PSA, after which we have named the states **D**efault (where there is a single winner) and near-**T**ie.

Our main result in this section is characterising the set of single-top PSAs that can be implemented with explicitly simple payments, while showing that multi-top PSAs can never be.

Since we already know from Sec. 4.1 that strategyproofness is characterized by indifference at the boundary/tie profiles, we need to:

1. Identify which boundary/tie profiles are possible in each class of PSAs;
2. Understand what constraints are imposed on the payments (and possibly on the PSA) by indifference at such profiles.

5.1 Identifying all Boundary and Tie Profiles

Each vertical arrow in position j in Fig. 3 means that agent j connects the two states (meaning there is a boundary profile where she is pivotal). We can see in the figure that all agents connect both states.

We say that agents $j, j+1$ are *adjacent* in state s if there is a profile $b \in \mathcal{B}(s)$ where they are tied (recall that j refers to the agent's rank).

Each horizontal arrow in Fig. 3 shows a pair $j, j+1$ that are adjacent in the respective state. We can see in the figure that all pairs are adjacent in both states, except possibly the pair $1, 2$ in state D. This will be the crucial difference between multi-top PSAs (where the edge always exist), and single-top PSAs, where the edge exists if and only if $\gamma \geq \alpha_2$. For proofs, see full version.

5.2 Indifference Constraints

Every arrow in the diagram shown in Fig. 3 imposes constraints on the payments, since it means that in any corresponding boundary profile (for a vertical edge) or tie profile (for a horizontal edge), the respective agent should be indifferent between the two states / ranks.

Lemma 4. *$u_j^{(T)}(v_j, b_{-j}) = u_j^{(D)}(v_j, b_{-j})$ in all profiles where j is pivotal between T and D, if and only if:*

$$\forall i < j, \;\; t_j^{(T)[i]} = \frac{1}{p_j^{(T)}} \left[p_j^{(D)} t_j^{(D)[i]} - \frac{\alpha_i}{\alpha_j}(p_j^{(T)} - p_j^{(D)}) \right] \tag{3}$$

$$\forall i \geq j, \quad t_j^{(T)[i]} = \frac{1}{p_j^{(T)}} \left[p_j^{(D)} t_j^{(D)[i]} - \frac{\alpha_{i+1}}{\alpha_j}(p_j^{(T)} - p_j^{(D)}) \right] \tag{4}$$

The proof idea is that by equating the utilities at the boundary profile, we get an expression of the form

$$\sum_{i \neq j} F_i(p_j^{(T)}, p_j^{(D)}, t_j^{(T)}, t_j^{(D)}, \bar{\alpha}) b_i = 0$$

for some functions $F_1, \ldots, F_{n-1}$. The only way to nullify the entire expression for a generic profile b is to make sure $F_i = 0$ for all i. The solution gives Eqs. (3),(4). The proof of the next lemma uses the same idea.

Lemma 5. *If $u_j^{(s)}(v_j, b_{-j}) = u_{j+1}^{(s)}(v_j, b_{-j})$ in all profiles $b \in \mathcal{B}(s)$ where $b_j = b_{j+1}$, if and only if:*

$$\forall i \neq j, \quad t_j^{(s)[i]} = t_{j+1}^{(s)[i]} \frac{p_{j+1}^{(s)}}{p_j^{(s)}} \tag{5}$$

$$t_j^{(s)[j]} = 1 + (t_{j+1}^{(s)[j]} - 1) \frac{p_{j+1}^{(s)}}{p_j^{(s)}} \tag{6}$$

This means that any arrow in the diagram completely locks the values of two weight vectors to one another. Since our arrow diagram contains cycles for any $n \geq 3$, this means that the values of t are overconstrained, and we get a contradiction unless the diagram commutes (i.e. when moving from $t_j^{(D)}$ to $t_{j+1}^{(T)}$ we get the same value whether we pick first the horizontal or the vertical edge.

Proposition 3. *The diagram in Fig. 3 commutes at $j, j+1$ if and only if*

$$\frac{\alpha_{j+1}}{\alpha_j} = \frac{p_{j+1}^{(T)} - p_{j+1}^{(D)}}{p_j^{(T)} - p_j^{(D)}}. \tag{7}$$

This means that not every PSA is allowed: setting the allocations $p^{(T)}$ and $p^{(D)}$ also dictates $\alpha_2, \ldots, \alpha_n$. Further, setting any $t_j^{(s)}$ uniquely determines all payments, if exist.

For multi-top PSAs, the allocations also dictate α_1 and we show in the full version that the induced cutoff between states is such that T is never realized (so there is only one state).

ALGORITHM 1: COMPUTE SINGLE-TOP TWO-STATE PAYMENT RULE

Input: Partial PSA $\bar{p} = (p^{(D)}, p^{(T)})$; Tie-sensitivity parameter $\gamma \leq p_2^{(T)} - p_2^{(D)}$.
Output: An explicit auction (x, PAY).
Set $\alpha_j \leftarrow p_j^{(T)} - p_j^{(D)}$ for all $j > 1$;
Set $\alpha_1 \leftarrow -\gamma$;
(If $\gamma = \alpha_2$ then set tie-breaking towards state T);
Initialize $t_n^{(D)[i]} \leftarrow 0$ for all $i \leq n-1$;
Set $t_n^{(T)}$ using $t_n^{(D)}$ and Eqs. (3), (4);
For each $j = n-1, n-2, \ldots, 1$ set $t_j^{(T)}$ using $t_{j+1}^{(T)}$ and Eqs. (5), (6);
For each $j = n-1, \ldots, 1$ set $t_j^{(D)}$ using $t_j^{(T)}$ and Eqs. (3), (4);
Return (x, PAY) (x is defined by $\bar{p}$ and $\bar{\alpha}$, PAY is defined by t);

5.3 Explicitly Simple Implementation for Single-Top PSA

We can summarize the result for single-top PSAs, which is our main positive result in the paper (for a full proof see full version):

Theorem 2. *A single-top PSA has an explicitly simple implementation if and only if Eq.* (7) *holds for all* $j > 1$.

Moreover, given any partial PSA $\bar{p} = (p^{(D)}, p^{(T)})$ *and a tie-sensitivity parameter* $0 \leq \gamma \leq p_2^{(T)} - p_2^{(D)}$*, Algorithm 1 returns an auction that is SP, IR and consistent with* $\bar{p}$ *and* γ*.*

The double arrows in Fig. 4 show the order in which Algorithm 1 sets the payments (although any order would do), and next to it the prices derived for the PSA in Example II. The reader can compare this succinct tabular form to the one derived for the Myerson payments in Fig. 2.

Proof sketch of the positive direction. As stated in Sec. 5.1, setting $-\alpha_1 = \gamma \leq p_2^{(T)} - p_2^{(D)}$ guarantees that adjacency and connectedness edges are as in Fig. 3, and in particular that there is no edge (constraint) between $t_1^{(D)}, t_2^{(D)}$.

The algorithm then guarantees (due to Lemmas 5 and 4) that the respective agents are indifferent in every boundary and tie profile corresponding to each edge (note that the edge corresponding to α_1 is not part of a cycle). This guarantees SP by Lemmas 2 and 3. IR follows from strategyproofness, as no bidder can gain by becoming last, and the last bidder is guaranteed a non-negative utility (since the payment is set to 0). □

Finally, an important question is whether our explicit implementation requires positive payments to agents. For single-top PSA, we provide a full answer.

Proposition 4. *Consider any SP single-top explicitly simple auction. Then either*

- $n = 2$*; or*
- $n = 3$ *and* $\gamma = \alpha_2$*; or*
- *IR is violated; or*
- *NPT is violated.*

$$\begin{array}{ccccccccc}
0 = t_n^{(D)} & & t_{n-1}^{(D)} & & t_j^{(D)} & & t_2^{(D)} & & t_1^{(D)} \\
\Downarrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
t_n^{(T)} & \Rightarrow & t_{n-1}^{(T)} & \cdots & t_j^{(T)} & \cdots & t_2^{(T)} & \Rightarrow & t_1^{(T)}
\end{array}$$

$t_j^{(s)}$	$s = D$	$s = T$
$j = 1$	$(11/8, 3/4) \Leftarrow$	$\Uparrow$ $(3/4, -1/2)$
$j = 2$	$(0, 0) \quad \Leftarrow$	$\Uparrow$ $(1/2, -1)$
$j = 3$	$(0, 0) \quad *$	$\Rightarrow$ $(1/2, -1)$

Fig. 4: Left: Adjacency and connectedness constraints in One-Top PSAs. The initialization point and the double arrows show the order of computation as preformed in Alg. 1.
Right: Explicitly simple payments for Example II, in tabular form.

$$\begin{array}{cccccccccc}
0 = t_n^{(|S|)} & \leftrightarrow & t_{n-1}^{(|S|)} & \leftrightarrow \cdots \leftrightarrow & t_j^{(|S|)} & \leftrightarrow \cdots \leftrightarrow & t_2^{(|S|)} & & t_1^{(|S|)} & \text{(Wide gap in bids)} \\
\Downarrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & \\
t_n^{(|S|-1)} & \leftrightarrow & t_{n-1}^{(|S|-1)} & \leftrightarrow \cdots \leftrightarrow & t_j^{(|S|-1)} & \leftrightarrow \cdots \leftrightarrow & t_2^{(|S|-1)} & & t_1^{(|S|-1)} & \\
\Downarrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & \\
\ldots & & & & & & & & & \\
\Downarrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & \\
t_n^{(1)} & \Rightarrow & t_{n-1}^{(1)} & \Rightarrow \cdots \Rightarrow & t_j^{(1)} & \Rightarrow \cdots \Rightarrow & t_2^{(1)} & \Rightarrow & t_1^{(1)} & \text{(Close to a tie)}
\end{array}$$

Fig. 5: Adjacency and connectedness constraints in Multi-state Ordered One-Top PSAs. The Double arrows show a possible order of computation.

This means that in the first two cases, an explicitly simple implementation will coincide with Myerson's payments. In contrast, in all other cases any explicitly simple implementation must *differ* from Myerson's payments, i.e. require some non-zero $p^0_{\underline{k}}$.

E.g. for the example in Fig. 1 we can verify that the explicitly simple implementation requires $p^0_{\underline{c}}((1,3)) = -\frac{1}{8}$.

In particular the proposition characterizes when Myerson's payments (with $p^0_{\underline{k}} \equiv 0$) are explicitly simple according to our definition. Note that we do not have a way to find the appropriate translation $p^0_{\underline{k}}$ (when exists), but we do not need one since we derived the explicit formulation directly.

5.4 Beyond the Two States Solution

Our results also extend to more than two states, as long as states are *ordered*, meaning that no three states share a boundary. See full version for details.

Intuitively, ordered states means that no profile is in the intersection of three states or more, e.g. when we can order states from 'closest to a tie' to 'farthest from a tie'. We then have a similar structure of constraints between adjacent weight vectors, which has a solution only when the allocation rule is single-top, breaking the cycles involving agent 1 (see Fig. 5). Note that this generalizes both two-agents and two-states cases.

6 Conclusion

We showed how certain natural allocation rules that diversify the set of winners when close to a tie, can be implemented using an explicit payment, where each agent simply pays a fixed combination of the other bids, which depends on her rank (and on whether a near-tie was declared). The designer has limited freedom: she can choose any two allocations for the two states, provided that monotonicity is maintained and that only the leading bidder loses some of the item under near-tie. Then there is only one more free parameter, which can be thought of as the 'sensitivity' of the allocation rule to near-ties. Setting this parameter uniquely defines the allocation rule and the payment.

While explicitly simple auctions for $n \geq 3$ require payments to some agents, we believe that their simple structure makes them preferable over basic (non-negative prices) Myerson payments in some situations, and that increasing buyer retention and diversity will pay off at the long run.

Natural followup directions are to study the conditions under which budget balance is guaranteed, and, more broadly, to study welfare and revenue implications when the distribution of buyer types is given. This would enable us to design good near-tie auctions that balance the different goals of the auctioneer.

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