Parity in Markets - Methods, Costs, and Consequences

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Abstract

Fisher markets are those where buyers with budgets compete for scarce items, a natural model for many real world markets including online advertising. We show how market designers can use taxes or subsidies in Fisher markets to ensure that market equilibrium outcomes fall within certain constraints. We adapt various types of fairness constraints proposed in existing literature to the market case and show who benefits and who loses from these constraints, as well as the extent to which properties of markets including Pareto optimality, envy-freeness, and incentive compatibility are preserved. We find that several prior proposed constraints applied to markets can hurt the groups they are intended to help.

1 Introduction

A market solves the economic problem of "who gets what and why" (Roth 2015). Market mechanisms are in place in many 'platform businesses' such as online advertising, ride-sharing, and others. Unfortunately, there is no guarantee that market outcomes will always align with other objectives of the market designer such as business needs or legal constraints. Recent work proposes the addition of 'fairness constraints' into allocation mechanisms (Celis, Mehrotra, and Vishnoi 2019; Ilvento, Jagadeesan, and Chawla 2020; Chawla and Jagadeesan 2022; Celli et al. 2022; Balseiro, Lu, and Mirrokni 2021). Our contribution to this problem two-fold. The first is technical: we show how market designers can use price interventions in a Fisher market to implement constraints on market outcomes. The second is conceptual, we show the importance of studying second-order effects of making these interventions as who wins and who loses from interventions may not follow the intentions of the intervention.

We focus on Fisher markets where budget constrained buyers compete for items that are in limited supply (Eisenberg and Gale 1959). Given prices for items, demand is the result of buyers maximizing their utilities. An equilibrium in Fisher markets are prices and allocations such that supply equals demand.

The Fisher model allows us to abstract away from the details of how a market works and study equilibria as properties of demand and supply directly. For example, even though individual impressions in many real world advertising markets are allocated via paced auctions, the *aggregate* outcome across all advertisers and impressions is that prices make supply meets demand - in other words, a Fisher market equilibrium (Conitzer et al. 2019, 2022).

From the perspectives of buyers and market designers, Fisher equilibria are well understood. The allocations are Pareto-optimal and envy-free up to budgets (no buyer strictly prefers another buyer's bundle to their own bundle when budgets are equal) (Varian 1974; Budish and Cantillon 2012). When markets are 'large' buyers have no incentives to lie about their valuations of items (Azevedo and Budish 2018; Peysakhovich and Kroer 2019).

However, equilibrium-based allocations can have undesirable distributive properties from the point of view of group-level fairness. For example, recent work argues advertising markets may result in outcomes which may be 'unfair' in various senses (Ali et al. 2019; Zhang 2021; Imana, Korolova, and Heidemann 2021).

We ask how market designers can 'adjust' market outcomes to fix these undesirable distributional properties.

A common tool for market designers is price intervention (Pigou 1924) and our main technical contribution is to give designers a way to target these in Fisher markets. Given a family of linear constraints, we show how to use convex optimization construct a set of price interventions - taxes and subsidies for some purchases - such that there are allocations which satisfy the constraints and are also market equilibria.

Many constraint families proposed in the literature can be written as linear constraints. For each of these proposed constraint families we ask: who wins and who loses when these interventions are made? The results are not always intuitive and, because second-order effects abound in markets, do not always achieve the stated goals of the constraints.

Motivated recent by literature (Ali et al. 2019: Celis, Mehrotra, and Vishnoi 2019; Ilvento, Jagadeesan, and Chawla 2020; Chawla and Jagadeesan 2022; Celli et al. 2022) we first consider buyer parity constraints where items are split into groups A and B and each buyer in a set is required to have parity in allocation with respect to these groups. This constraint guarantees parity in exposure between the A and B groups, however it does not guarantee Pareto optimality even when item group welfare is taken into account. In

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addition, this intervention can lead to the previously disadvantaged group receiving *less* aggregate exposure than before.

Motivated by the work on fairness in applicant/recruiter job search matching (Geyik, Ambler, and Kenthapadi 2019) we also consider the 'transpose' of the above constraint - per item parity constraints. Buyers have group labels A and Band some set of items are each required to have equal exposure in the buyer groups. Here the goal is that items (e.g. recruiter impressions) have parity in exposure to buyers (e.g. resumes of two groups of individuals). Here, again, we lack Pareto optimality and the intervention can reduce the utility of a previously disadvantaged buyer group.

We consider a 'floor' constraint where we require that a subset of buyers have a minimum exposure to a subset of items. We find that this intervention gives outcomes that are Pareto optimal if item exposure utility is taken into account but not otherwise. Satisfying this constraint in market outcomes is equivalent to the market designer subsidizing constrained buyers' purchases of the protected items, and though this may sound uniformly welfare improving, we show that buyers in the constrained group can have their welfare reduced by the intervention.

Finally, we show that each of the above constraint families maintains the incentive properties of Fisher markets as long as the group of buyers on whom the constraint applies grows large with the market.

2 Related Work

There is large interest in centralized market allocation as a solution to the multi-unit assignment problem (Budish and Cantillon 2012). Here individuals report their valuations for items, a center computes an (approximate) equal-budget equilibrium allocation, and returns this allocation to the agents. This mechanism is known as competitive equilibrium from equal incomes (CEEI, (Varian 1974)) and is used in the allocation of courses to university students though with more complex utility functions and computations than our Fisher case (Budish et al. 2013, 2016). Recent work studies how general constraints can be added to CEEI allocations (Echenique, Miralles, and Zhang 2021). Our work adds to this literature by showing properties of specific constraints proposed in discussions of demographic fairness in allocation. We do not seek to prove general results about all possible constraint families. Instead, we focus on the more restrictive case of linear constraints in Fisher markets which allows us to derive stronger results-i.e. the convex program to determine optimal taxes/subsidies-than in general unit demand markets studied by Echenique, Miralles, and Zhang (2021).

Market equilibria in Fisher markets and their relationship to convex programming are well studied (Eisenberg and Gale 1959; Shmyrev 2009; Cole et al. 2017; Kroer et al. 2019; Cole et al. 2017; Cole and Gkatzelis 2018; Caragiannis et al. 2016; Murray et al. 2019; Gao and Kroer 2020). Our work complements this existing work as we show that the convex program equivalence allows us to easily construct market interventions in the forms of taxes and subsidies.

A recent literature has begun to study differential outcomes in online systems (Ali et al. 2019: Gevik, Ambler, and Kenthapadi 2019; Zhang 2021; Imana, Korolova, and Heidemann 2021) across categories of users (e.g. job ads across gender). Though a highly studied cause of these differential outcomes are biases in machine learning systems, there are also market forces that can cause such issues. For example, Lambrecht and Tucker (2018) studies STEM job ads competing in a real ad market and shows that "younger women are a prized demographic and are more expensive to show ads to. An algorithm that simply optimizes cost-effectiveness in ad delivery will deliver ads that were intended to be gender neutral in an apparently discriminatory way, because of crowding out." Our work further highlights the importance of understanding market dynamics before committing to various interventions.

Other work has begun to study unintended consequences of fairness interventions (Fang and Moro 2011; Liu et al. 2018; Emelianov et al. 2022) as well as the fact that in many cases machine learning is used in ways where there are strategic agents (Hardt et al. 2016; Hu, Immorlica, and Vaughan 2019). The market setting is particular interesting for both of these literatures as markets are places with large second order effects as well as strategic agents.

А recent literature has looked at impleof menting various notions fairness in single auctions (Celis, Mehrotra, and Vishnoi 2019; Ilvento, Jagadeesan, and Chawla 2020; Chawla and Jagadeesan 2022), paced auctions (Celli et al. 2022) online allocation or problems (Balseiro, Lu, and Mirrokni 2021). Our work complements this by showing that in markets similar implementation can be done via the use of the price system. In addition, these papers study the performance of their algorithms but bypass the questions such as second order market effects, who wins and who loses, and whether good incentive properties are preserved.

3 Fisher Markets

Consider a market where there are n buyers (e.g. advertisers) and m items (e.g. individuals' ad slots). Each item has supply s_j which for the purposes of this section we take to be 1. We assume that fractional allocations are allowed (in ad auctions these can be thought of as randomized allocations). Fractional allocation ensures existence and polynomial time computability of equilibria. In practice Kroer and Peysakhovich (2019) shows numerically that in many Fisher markets with linear utility, almost all assignments are 1 or 0.

For an allocation $x \in \mathbb{R}^{n \times m}_+$, we let $x_i \in \mathbb{R}^m_+$ be the bundle of goods assigned to buyer *i* with x_{ij} being the assignment of item *j*. Each buyer has a utility function $v_i(x_i)$.

We assume the utilities are all homogenous of degree one (i.e. that $\alpha v_i(x_i) = v_i(\alpha x_i)$ for $\alpha > 0$), concave, and continuous. We also assume that there exists an allocation x such that $v_i(x_i) > 0$ for all buyers i. This class of utilities includes the constant elasticity of substitution (CES) family of utilities, and linear utilites are special case of CES utilities where each buyer has a value v_{ij} for each item and $v_i(x_i) = \sum_j v_{ij}x_{ij}$. Linear utilities are precisely what is assumed in many studies of online advertising markets (Conitzer et al. 2019; Balseiro et al. 2021).

Each item will be assigned a price $p_j > 0$, with p being the full price vector, and each buyer has a budget $B_i > 0$. We study the quasi-linear (QL) case where leftover money is kept by the buyer; this is most natural for several real world markets such as advertising markets. Most of our results work the same for the non-QL case.

Let δ_i be the leftover budget under prices p with allocation x_i ($\delta_i = B_i - p^T x_i$). The quasilinear utility that a buyer experiences under prices p and allocation x is then

$$u_i(x_i, \delta_i) = v_i(x_i) + \delta_i$$

Given a price vector *p*, the **demand** of a buyer is

$$D_i(p) = \operatorname*{argmax}_{x_i:x_i^\top p \le B_i} u_i(x_i, \delta_i).$$

A market equilibrium is an allocation x^* and a price vector p^* such that 1) allocations are demands: for each i $x_i^* \in D_i(p^*)$, and 2) markets clear: for each j, $\sum_i x_{ij} \leq 1$, with equality if $p_j > 0$.

In Fisher markets with our family of utilities, market equilibria always exist. The general problem of market equilbrium is computationally difficult (Chen and Teng 2009; Vazirani and Yannakakis 2011; Chen, Kroer, and Kumar 2021) but in the family of utilities we are considering the equilibrium allocation can be found via solving a version of the Eisenberg-Gale convex program (EG) (Eisenberg and Gale 1959):

$$\max_{\substack{x \ge 0, \delta \ge 0}} \sum_{i} B_{i} \log(v_{i}(x_{i}) + \delta_{i}) - \delta_{i}$$

s.t.
$$\sum_{i} x_{ij} \le 1, \forall j = 1, \dots, m$$
 (1)

The objective maximizes a monotone transformation of the budget-weighted geometric mean of buyer utilities. We can get the price vector for the market equilibrium allocation¹ by looking at the values of the Lagrange multipliers on the supply constraints at the optimum (Eisenberg and Gale 1959).²

Desirable Properties of Market Equilibria

Equilibria have several desirable properties: first, they are Pareto efficient. In the QL case this is defined using the leftover budget as well as the seller who receives the payments. Let (x_i, δ_i) be the bundle of goods received by buyer i and their leftover budget with (x, δ) being all buyers' allocations/budgets. An allocation (x, δ) is **Pareto optimal** if for every alternative allocation (x', δ') such that some buyer strictly improves their utility, it must be the case that for some other buyer i, we have $v_i(x'_i) + \delta'_i < v_i(x_i) + \delta_i$ or $\sum_i \delta'_i > \sum_i \delta_i$, meaning that the seller is worse off. Similarly, if $\sum_i \delta'_i < \sum_i \delta_i$, meaning the seller strictly improves, then it must be the case that for some buyer i, we have $v_i(x'_i) + \delta'_i < v_i(x_i) + \delta_i$.

Equilibrium allocations are **envy free** when accounting for budgets. Let (x, δ) be an equilibrium allocation. For any two buyers i, i' with budget ratio $\gamma = \frac{B_i}{B_{i'}}$ we always have that $v_i(x_i) + \delta_i \ge v_i(\gamma x_{i'}) + \gamma \delta_{i'}$.

Envy freeness is related to another important property: strategy-proofness in the large. We consider 'large' in the market context follows: let there be n buyers, each with budget 1. The size of the market is determined by this n. Let there be m items with generic item j, and let the supply of each item be s_jn where s_j is some constant.

Consider the *Fisher mechanism*: each individual reports their type (i.e. their valuation function) t_i , a center computes the market equilibrium, and gives each individual their corresponding allocation. At a high level, a mechanism is said to be **strategy-proof in the large (SPL)** if for any $\epsilon > 0$ for all types t_i , there exists \bar{n} such that if $n > \bar{n}$ the gain to any buyer of type t_i from misreporting any t'_i instead of their true t_i is less than ϵ .³

In Appendix C, we give a formal definition as well as expand this definition to cover the constrained case.

Importantly though, it is known that:

Known Properties 1. *The centralized Fisher mechanism is SPL.*

All of these properties are important reasons that markets are considered useful allocative mechanisms.

4 Changing Market Outcomes

We consider a market designer that wants market allocations to satisfy certain constraints. Though in some cases (e.g. fully centralized) the market designer can control quantities allocated directly, in some cases the market designer must work through the price system, or they may wish to do so due to efficiency properties derived from doing so.

We consider a market designer that is able to subsidize and tax purchases. Let \bar{p} be an $n \times m$ matrix of these price interventions. We assume that each buyer *i* faces price $p_j + \bar{p}_{ij}$ for item *j* with $\bar{p}_{ij} > 0$ being a tax and $\bar{p}_{ij} < 0$ being

¹There may be multiple equilibria in a Fisher market, however the multiplicity only exists 'up to ties' in the allocation, in the sense that in all equilibria the prices are the same, and though allocations may differ, the utility realized by each buyer is the same. The intuition for this uniqueness comes from the convex program equivalence – though there may be multiple solutions, the program is strictly convex in utilities, which leads to uniqueness.

²The fact that Fisher equilibria are allocations that maximize the budget weighted product of utilities is an important reason they are studied in the fair division community. In the equal budget case Nash social welfare has been called an "unreasonably effective fairness criterion" (Caragiannis et al. 2016) and is used in practice in allocation mechanisms such as Spliddit.com (Goldman and Procaccia 2014).

³In the real world, auction-based ad markets act like a semicentralized Fisher mechanism. Buyers report their valuations for various events (clicks, conversions, impressions, etc...) and the ad system bids for them for each impression. In this case, a notion of strategy proofness is extremely important.

a subsidy. Let $\bar{p}_i \in \mathbb{R}^m$ be the vector of price interventions for buyer *i*.

The demand of a buyer i with base prices p and interventions \bar{p} is then $D_i(p+\bar{p}_i)$. We can now extend the definition of a market equilibrium. We say that given a \bar{p} the triple (x, p, \overline{p}) is a **tax-subsidy equilibrium** if

1. Each
$$x_i \in D_i(p + \bar{p}_i)$$

2. For all $j \sum_{i} x_{ij} \leq 1$, with equality if $p_j + \bar{p}_{ij} > 0$.

Now, suppose that the market designer wants to force the market equilibrium outcome to satisfy a certain set of constraints by appropriately designing a price intervention. That is, we look for a system of taxes and subsidies \bar{p} such that there exists (x^*, p^*) such that (x^*, p^*, \overline{p}) is a tax-subsidy equilibrium. The next proposition shows how to construct such a \bar{p} :

Proposition 1. Let $\mathcal{D} = \{x : A_1x \leq b_1, A_2x = b_2\}$ for $A_1 \in \mathbb{R}^{K_1 \times m}, A_2 \in \mathbb{R}^{K_2 \times m}, b_1 \in \mathbb{R}^{K_1}, b_2 \in \mathbb{R}^{K_2}$ be a set of linear constraints that the market designer wants an equilibrium allocation to satisfy. Suppose there exists a supplyfeasible allocation x such that the constraints are satisfied and $v_i(x_i) > 0$ for all i. Then we can compute, in polynomial time, a tax/subsidy \bar{p} along with a tax-subsidy equilibrium (x, p, \bar{p}) such that all of the linear constraints are satisfied.

The proof of this proposition is instructive as it provides the algorithm for computing \bar{p} from a set of constraints so we leave a high-level version of the proof in the text.

Proof. The proof is based on a straightforward generalization of the argument for the Eisenberg-Gale convex program with the additional constraint that $x \in \mathcal{D}$:

$$\max_{\substack{x \ge 0, \delta \ge 0}} \sum_{\substack{i \in T \\ i \in T}} B_i \log \left(v_i(x_i) + \delta_i \right) - \delta_i$$

s.t.
$$\sum_{\substack{i \in T \\ i \in T \\ i \neq i}} x_{ij} \le 1, \quad \forall j,$$
$$A_1 x \le b_1,$$
$$A_2 x = b_2.$$
 (2)

Recall that in the standard EG argument, the prices correspond to the Lagrange multipliers $\{p_j\}_{j \in [m]}$ on the supply constraints at the optimum.

Here, we end up with an additional Lagrange multiplier

There, we can up with an additional Lagrange multiplier $\lambda_1 \in \mathbb{R}_+^{K_1}$ and $\lambda_2 \in \mathbb{R}_+^{K_2}$. To construct a tax-subsidy equilibrium we construct price interventions $\bar{p}_{ij}^* = \sum_{k=1}^{K_1} A_{1kj} \lambda_{1k}^* + \sum_{k=1}^{K_2} A_{2kj} \lambda_{2k}^*$ using the Lagrange multipliers at optimality, set prices equal to the Lagrange multipliers p_i^* on the supply constraints, and take the optimal solution x^* of Eq. (2) as the corresponding allocation.

We extend the EG argument for why the corresponding x^*, p^* from Eq. (1) constitute a market equilibrium, we use KKT conditions as well as a generalization of Euler's identity for homogeneous functions to show that $(x^*, p^*, \overline{p}^*)$ form a tax-subsidy equilibrium.

Let x, δ be an optimal solution to Eq. (2). Such a solution exists by our assumptions. We use $\nabla_i u_i(x_i, \delta_i)$ to denote the j'th component of an arbitrarily-selected subgradient of $v_i(x_i)$. Let ν_i and μ_{ij} be the Lagrange multipliers for $\delta_i \ge 0$ and $x_{ij} \ge 0$ respectively.

We start by showing that each buyer i spends their budget exactly. We let $u_i = u_i(x_i, \delta_i)$. Now consider the KKT conditions for Eq. (2) (we leave out primal and dual feasibility conditions here since they are straightforward):

1. (Stationarity) For each x_{ij} we have

$$(\nabla_j u_i(x_i, \delta_i)) \frac{B_i}{u_i} - p_j - \sum_{k=1}^{K_1} A_{1kj} \lambda_{1k} - \sum_{k=1}^{K_2} A_{2kj} \lambda_{2k} + \mu_{ij} = 0,$$

and for δ_i we have $\frac{B_i}{u_i} + \nu_i = 1$. 2. (Complementary slackness)

$$x_{ij}\mu_{ij} = 0, \forall i, j, \tag{3}$$

$$\delta_i \nu_i = 0, \forall i, \tag{4}$$

$$p_j(1-\sum_j x_{ij}) = 0, \forall j, \tag{5}$$

$$\lambda_1^{\top}(b_1 - A_1 x) = 0, \tag{6}$$

$$\lambda_2^{+}(b_2 - A_2 x) = 0. \tag{7}$$

Rewriting stationarity, using the fact that $x_{ij}\mu_{ij} = 0$, and using our definition of \bar{p}_{ij} , we get

$$(\nabla_j u_i(x_i, \delta_i))\frac{B_i}{u_i} \le p_j + \bar{p}_{ij},$$

where equality holds if $x_{ij} > 0$. Let $\tilde{p}_{ij} = p_j + \bar{p}_{ij}$. Now we multiply each side of the rewritten stationarity condition by x_{ij} , sum over j, and add $\delta_i \frac{B_i}{u_i} = \delta_i$ to get

$$\sum_{j} x_{ij} (\nabla_j u_i(x_i, \delta_i)) \frac{B_i}{u_i} + \delta_i \frac{B_i}{u_i} = \sum_{j} x_{ij} \tilde{p}_{ij} + \delta_i.$$

Now we can apply the generalized Euler identity for subd*ifferentials* (Yang and Wei 2008): let $q \in \partial v_i(x_i)$ belong to the subdifferential of v_i at x_i . Then the generalized Euler identity states that $\sum_j g_j x_{ij} = v_i(x_i)$.

Applying this identity we get

$$\sum_{j} x_{ij} \tilde{p}_{ij} + \delta_i = u_i \frac{B_i}{u_i} = B_i.$$

Since the left-hand side is expenditure, we get that buyer ispends their budget exactly.

Next, we need to show that for every alternative bundle x'_i, δ'_i such that $\sum_j x'_i \tilde{p}_{ij} + \delta'_i \leq B_i$, we have $u_i \geq u_i(x'_i, \delta'_i)$. By concavity of u_i we have

$$u_{i}(x'_{i}, \delta'_{i}) - u_{i} \leq \sum_{j} \nabla_{j} u_{i}(x_{i}, \delta_{i})(x'_{ij} - x_{ij}) + \delta'_{i} - \delta_{i}$$

$$= \frac{u_{i}}{B_{i}} \sum_{j} (\tilde{p}_{ij} - \mu_{ij})(x'_{ij} - x_{ij}) + \frac{u_{i}}{B_{i}} (1 - \nu_{i})(\delta'_{i} - \delta_{i})$$

$$= \frac{u_{i}}{B_{i}} (\sum_{j} (\tilde{p}_{ij} - \mu_{ij})x'_{ij} + (1 - \nu_{i})\delta'_{i} - B_{i})$$

$$\leq \frac{u_{i}}{B_{i}} (\sum_{j} \tilde{p}_{ij}x'_{ij} + \delta'_{i} - B_{i})$$

$$\leq 0$$

where the first equality follows by stationarity, and the second by complementary slackness and the fact that (x_i, δ_i) spends the budget exactly. The second-to-last inequality follows because all variables are positive, and the last inequality follows by budget feasibility. Thus we have shown that each buyer *i* receives a bundle x_i belonging to their demand set $D_i(p + \bar{p}_i)$.

Finally we need to check the market clearing condition: this follows immediately from complementary slackness on p_j and primal feasibility.

We also get a straightforward corollary about the power of taxes/subsidies in a Fisher market.

Corollary 1. For any feasible allocation \bar{x} , we can use a convex program to compute a price intervention \bar{p} such that there is a tax-subsidy equilibrium (\bar{x}, p, \bar{p}) .

The corollary above follows from the fact that $\{x_{ij} = \bar{x}_{ij}\}$ is a family of linear constraints. Of course, for "extreme" allocations the market designer may be required to make large taxes or subsidies.

Note that arbitrary systems of constraints may require a complex system of taxes/subsidies (e.g. specific taxes/subsidies for each buyer/item pair).

However, the "simpler" the constraint, the simpler the tax/subsidy required to attain it. For example, consider the *exposure floor* constraint that we will study further in later sections. Here B is a subset of buyers and R is a subset of items and our goal is that in aggregate buyers in B have at least some fixed level of exposure to R. This is written as the constraint $\sum_{i \in C} \sum_{j \in B} x_{ij} \ge L$.

A tax/subsidy to make our equilibrium allocation satisfy this goal will only require one common subsidy \overline{P} that all buyers in B will face for all items in R.

The effects of taxes and subsidies in Fisher markets are not always straightforward. In particular, in common discussion taxes are viewed as ways to decrease demand (and increase price) and subsidies for the opposite. However, this is more complex in Fisher markets because "base" prices p are determined in equilibrium **given** the market designer chosen tax/subsidy schedule \bar{p} .

Consider what happens when the market designer subsidizes all purchases by all individuals an equal amount \tilde{p} . Here new equilibrium 'base' prices will simply be the original equilibrium prices increased by \tilde{p} and there will be no net effect on revenue or allocation. Similarly, in a 2 good market a blanket subsidy for good 1 is equivalent in effect to a blanket tax on good 2. Thus, while we continue to refer to price interventions as taxes and subsidies it is important to remember that it is relative, not absolute, levels of interventions that matter for outcomes.

5 Market Fairness Constraints

We now focus specifically on fairness constraints that have been brought up in existing literature.

Many of our motivating examples are from the literature on online ad markets. In this literature, the most commonly used utility function is the linear utility where buyers have vectors of utilities for each item and the total utility of a bundle is the sum of the item utilities $v_i(x_i) = v_i^T x_i$. To keep things simple we use this for our examples going forward.

We ask whether good properties of market allocations (Pareto optimality, envy freeness, SPL) are preserved under these interventions. In addition, we ask who wins and who loses from these interventions.

Per Buyer Fractional Parity

Ali et al. (2019) consider the issue of preferential exposure of certain groups to certain kinds of job ads. A similar issue occurs in advertising for certain legally protected categories (e.g. housing).

This leads to our first family, **per buyer fractional parity** (PBFP) constraints.

Let items have a binary attribute, either A or B. There is a subset of buyers called **constrained buyers** the set of which we denote by C. In order to combat disparate outcomes, constrained buyers are required to have 'balanced' exposure to items (e.g., in an online ad market these buyers could be housing or job ads). This leads to the following linear constraints: for any $i \in C$ we require that

$$\sum_{j \in A} x_{ij} = \alpha \sum_{j \in B} x_{ij}$$

Other examples in the literature can also be expressed as choices of α . For example, the equal exposure constraint sets $\alpha = \frac{N_B}{N_A}$ where N_K is the size of each group.⁴

For this section, we will focus on purely equal exposure for simplicity of exposition. We know that an equilibrium can be constructed to satisfy this constraint.

Importantly, the implicit or explicit goal of PBFP constraints is generally to help some underlying disadvantaged group. Therefore, any notion of Pareto optimality or winning and losing must also include this group.

Formally, given an allocation X. For a buyer i let the idisadvantaged item group be the one that has less exposure. Let the aggregate disadvantaged item group be the one that has less exposure to C buyers. We will also refer to this group as the **protected** item group.

Let P be a subset of items called the protected group. Let $\sum_{i \in C} \sum_{j \in P} x_{ij}$ be the **aggregate exposure of the protected group to the** C **buyers**. An allocation (x, δ) is **buyer-protected-item Pareto optimal** if for every alternative allocation (x', δ') such that some buyer strictly improves their utility or such that the protected group P gains more aggregate exposure to C buyers, it must be the case that for some other buyer i, we have $v_i(x'_i) + \delta'_i < v_i(x_i) + \delta_i$ or $\sum_i \delta'_i > \sum_i \delta_i$, meaning that the seller is worse off. Similarly, if $\sum_i \delta'_i < \sum_i \delta_i$, meaning the seller strictly improves, then it must be the case that for some buyer i, we have $v_i(x'_i) + \delta'_i < v_i(x_i) + \delta_i$ or the protected group loses exposure to C buyers. Finally, if P gets more aggregate exposure to C buyers it must be that either some buyer or the seller

⁴For the context of online job ads Ali et al. (2019) also suggest using a weight of number of expected qualified candidates in each group rather than simply group size.

V			X^{EG}					X^{PBFP}				
Buyer	Item A	Item B	Buyer	Item A	Item B	$B_i - px_i$	u_i	Buyer	Item A	Item B	$B_i - px_i$	u_i
C	1.5	.4	С	.33	0	.5	1	С	0	0	1	1
С	.4	1.5	С	0	.33	.5	1	С	0	0	1	1
U	5	2	U	.66	0	0	3.33	U	1	0	0	5
U	2	5	U	0	.66	0	3.33	U	0	1	0	5
			Price	1.5	1.5	-		Price	1	1	-	

Table 1: An example of a set of valuations with budgets set to 1 where imposing PBFP constraints leads to both parity constrained (C) buyers exiting the market completely and all items going to the unconstrained (U) buyers. Under the constraints, the U buyers get both items and get them cheaper than in the original equilibrium, so they are better off.

is strictly worse off. We refer to the original definition in Section 3 as **buyer-only Pareto optimality**.

We now ask whether PBFP constraints keep good market properties.

Proposition 2. *PBFP constrained equilibria are not buyeronly Pareto optimal. PBFP constrained equilibria are also not buyer-protected-item-group Pareto optimal.*

While lack of buyer-only Pareto optimality may not be surprising (since the goal of the intervention is to also improve the utility of a supply-group), the lack of buyerprotected-item-group Pareto optimality (shown in the Appendix Example Table 5) is more troubling.

In addition, imposing parity constraints on a single buyer also imposes pecuniary externalities (i.e. second order effects) on other buyers. The net of these second order effects may be large. This means, winners and losers are not clear.

Consider two buyers i, k. Suppose we add parity constraints only to i, then i is made worse off. However, the addition of parity constraints changes i's demand, it reduces demand for i's originally advantaged group and increases it for i's originally disadvantaged group. This means the second order effect can be positive or negative from buyer k's perspective. When implementing parity constraints on the entire group of C buyers, these second order effects can dominate, making net effects not obvious. In particular, the constraint may completely miss the mark of its original goal.

Proposition 3. Adding PBFP constraints can decrease the exposure of disadvantaged item groups to C buyers relative to the original equilibrium.

We see this in Example Table 1 adding parity constraints leads constrained buyers to exit the market. This means after the imposition of PBFP constraints *all items receive less exposure* to the constrained buyers. However, parity constraints also decrease competition and thus prices paid by unconstrained buyers.

In the context of the job advertising example, we added parity requirements to job ads leading to a particular type of parity - nobody receives any job ads. The only welfare improved is the 'regular' advertisers who now face less competition.

Note that even which group of buyers wins or loses is not guaranteed. For example, all buyers are worse off in the example in Appendix Table 4.

Shifting focus to envy-free and SPL properties, it is clear that in general there may be envy between a constrained and an unconstrained buyer who share the same valuation function (since the constrained buyer effectively has a strictly smaller choice set). However, within group this is not the case:

Proposition 4. If buyers i, i' are both constrained (or both unconstrained) then i does not have (budget adjusted) envy for i'.

Finally, and strongly related to the envy-free property, we see that SPL continues to hold:

Proposition 5. The Fisher market with PBFP constraints and constant fraction γ of constrained buyers who cannot misreport whether they are constrained is SPL.

We relegate proofs of both propositions to the Appendix.

Per Item Fractional Parity

We now turn to the 'transpose' of the PBFP constraint. Geyik, Ambler, and Kenthapadi (2019) consider the case of job candidate search and the goal of creating a balanced slate of candidates in any search query.

The market version of such a constraint is that items (e.g. impressions) have a balanced exposure to different types of buyers. For example, this can be used in certain recommender systems to balance the exposure of content consumers on certain types of content producers (Singh and Joachims 2018; Geyik, Ambler, and Kenthapadi 2019).

We refer to such constraints as per item fractional parity constraints (PIFP).

Formally, buyers are one of two types, either A or B. There is a subset of items which are constrained which we denote by C. Formally the constraint is given by

$$\forall j \in C, \ \sum_{i \in A} x_{ij} = \alpha \sum_{i \in B} x_{ij}.$$

As above this can always be expanded to different item groups C_1, C_2, \ldots with their own values of α .

As before, there are many choices of α . For simplicity, we again focus on parity of exposure, i.e. $\alpha = \frac{N_A}{N_B}$. Note though, that various other choices of α in the buyer parity case often have natural equivalents in the item parity case.

Given an allocation X, we call the buyer group G which has less exposure to items C the **disadvantaged buyer group**. With this in mind, we now focus on the welfare consequences of implementing per item fractional parity constraints.

V			X^{EG}					X^{PIFP} or X^A	EF			
Buyer A_1	Item C	Item U	Buyer A 1	Item C .167	Item U 0	$B_i - px_i$.66	u_i	Buyer A ₁	Item C .382	Item U	$B_i - px_i$	u_i
A_2	2	1.5	A_2	0	.75	0	1.12	A_2	.118	.42	.14	1
$B_1 \\ B_2$	3 3	$\frac{2}{2}$	$B_1 \\ B_2$.417 .417	.125 .125	0 0	1.5 1.5	$B_1 \\ B_2$.25 .25	.29 .29	0 0	1.33 1.33

Table 2: An example of a set of valuations with budgets set to 1 where all buyers prefer C to U and A buyers are disadvantaged originally. However, adding PIFP constraints fails to improve their utility and decreases the utility of A_2 . Note that this is equivalent an AEF constraint of requiring .5 exposure of C to A buyers.

Proposition 6. *PIFP constraints do not lead to buyer-only Pareto optimal allocations. Originally disadvantaged buyers may be worse off after constraints are implemented.*

To put this result into context, consider the use of something like CEEI for job recommendation for recruiters. Here each item is a recruiter impression and each buyer is a job applicant. This means we take a 'job applicant'-centric perspective, i.e., we focus on maximizing the applicant's goals with a Nash social welfare criterion as a distributional goal. On the other hand, to avoid discrimination on the recruiter side, we enforce parity of exposure to each recruiter exposure across some binary attribute of the applicants. Then, it is possible that the job applicants we are trying to help are worse off after this 'anti-discrimination' intervention.

It is easy to construct an example where all buyers are worse off under PIFP by simply considering that PIFP can force an equal split of items which clearly can be Pareto dominated by many allocations (see Table 3 in the Appendix).

However, PIFP can backfire in a different way as we see in Example Table 2. Here *all* buyers prefer item C to item U, and A buyers are less exposed to it than B buyers in equilibrium. Implementing PIFP fails to help A buyers. While it increases exposure of A_1 and A_2 buyers to C it strictly decreases the welfare of buyer A_2 because of increased competition from B buyers for item U.

As with the PBFP constraints we see that envy free and SPL properties continue to hold.

Proposition 7. If buyers i, i' are both A (or both B) then i does not have (budget adjusted) envy for i' in PIFP constrained equilibrium.

Proposition 8. The Fisher market with PIFP constraints and a constant fraction γ of buyers as A buyers where no buyers can misreport their group affiliation is SPL.

The full proofs of these propositions are in the Appendix.

Aggregate Exposure Floor Constraints

We now turn to studying an extremely simple constraint: we set a floor below which aggregate exposure of some buyers to some items cannot fall. We let C be a subset of constrained buyers and P a subset of protected items.

The **aggregate exposure floor constraint** (AEF) is written as

$$\sum_{i \in \mathcal{C}} \sum_{j \in P} x_{ij} \ge L$$

where L is our floor level. We require that L is jointly feasible with the supply constraints. When P is a singleton buyer, this can be thought of as a minimum quota.

We see that here at least some form of Pareto optimality is preserved:

Proposition 9. Adding AEF constraints to EG guarantees buyer-protected-item Pareto optimality. However, it does not guarantee buyer-only Pareto optimality.

The proof and counter-example for this proposition is can be found in the Appendix.

The AEF constraint can be used as an alternative to both PBFP and PIFP constraints above. Proposition 9 shows that it is more natural to wield the AEF constraint when the market designer cares specifically about the increasing the exposure of some set of items.

However, we can also consider the buyer-only case where there may be a disadvantaged buyer group that the designer cares about helping. For example, in the job applicant setting the designer may want to increase exposure of some applicant group (buyers) to some group of recruiters (items).

The AEF constraint is implemented via pricing by a blanket subsidy of \bar{p} for all buyers in C for all items in P. It seems intuitive that subsidizing buyers can only improve their utility, but this is actually incorrect:

Proposition 10. Adding AEF constraints can decrease the utility of buyers in the constrained group.

This follows from the fact that AEF constraints set with the right threshold can simulate PIFP constraints. The PIFP constrained equilibrium in Table 2 is also achievable with AEF constraints. Again, adding AEF constraints helps buyer C_1 get more of item P but buyer C_2 prefers the nonprotected item and, after implementation of AEF constraints, faces increased competition from buyers outside of the C set.

Thus, using AEF constraints purely as a way to improve buyer outcomes can also backfire.

Proposition 11. The Fisher market with AEF constraints is SPL when no buyers can misreport their group affiliation is per-group envy free and SPL.

Again, we relegate the proofs to the Appendix. Note that our SPL results require that the size of the constrained group grows to infinity as the market gets large - this means that the subsidy is done at the buyer/item group level and is, in the infinite limit, not affected by the report of an individual buyer. We cannot in general derive SPL results for individualized quota constraints per buyer since this would imply individualized subisidies and would incentivize buyers to underreport their valuation in order to increase their subsidy.

6 Conclusion and Future Directions

We have studied the implementation of parity constraints via prices. More importantly, we studied the effects of parity constraints and the winners and losers from their implementation.

We do not make claim that one type of intervention is *always* superior to another. The choice of intervention will, of course, be specific to the market being studied, the goals of the designer, and the sensitivity to various types of tradeoffs. Rather, we hope that our work shows that in market settings there can be many second order effects which make interventions work differently from what is intended or expected and these should be carefully studied and understood.

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A Appendix

B Proof of Results

We organize the section by equilibrium property.

Pareto Optimality

Counter Examples for PBFP Pareto Optimality. See Example 5 in the Extra Examples section.

Counter Example to AEF Buyer-Only Pareto Optimality. See Example 6 in the Extra Examples section.

Proof that AEF is Buyer-Protected-Item Pareto Optimal.

Let x^* be the optimal solution to the standard EG program, and let x^f be the optimal solution to EG with the floor constraint added for some set of buyers C and set of protected items A. We break the proof into two exhaustive cases.

Case 1: suppose that the EG objective is the same under x^* and x^f . In that case, it must be that $u_i(x_i^*, \delta_i^*) = u_i(x_i^f, \delta_i^f)$ for all buyers *i*, since there is a unique set of equilibrium utilities for EG (this follow from strict convexity of the log function). It then follows that x^f must be Pareto optimal, since x^* is Pareto optimal.

Case 2: suppose that the EG objective strictly decreases. In that case, let λ be the Lagrange multiplier on the floor constraint. Since the floor constraint leads to a strict decrease in objective, we must have that $\lambda > 0$. Now, by optimality and Lagrangian duality, we know that x^f must maximize the objective $\sum_i B_i \log u_i(x_i, \delta_i) + \lambda \sum_{i \in \mathcal{C}} \sum_{j \in A} x_{ij}$ over the set of supply-feasible allocations x. But then it follows immediately that x^f must be buyer-protected-item Pareto optimal, since if there exists an alternative allocation x such that $u_i(x_i, \delta_i) \geq u_i(x_i^f, \delta_i^f)$ and $\sum_{i \in \mathcal{C}} \sum_{j \in A} x_{ij} \geq \sum_{i \in \mathcal{C}} \sum_{j \in A} x_{ij}^f$, with strict inequality in at least one of these inequalities, then we could strictly improve the Lagrangian objective, which is a contradiction.

Since the EG objective must either stay the same or decrease under x^f , this shows that buyer-protected-item Pareto optimality holds.

Envy Freeness

In this section we show that allocation according to EG with PBFP, PIFP, or AEF constraints leads to envy-freeness within each buyer group. The proofs will all use the following observation:

Fact 1. For any pair of buyers i, i' in a market equilibrium (possibly with constraints) we have that if buyer i can afford $(B_i/B_{i'})(x_{i'} + \delta_{i'})$ under their personalized prices $p + \bar{p}_i$, then $u_i(x_i, \delta_i) \ge (B_i/B_{i'})u_i(x_{i'}, \delta_{i'})$.

This follows from the fact that buyer i receives something in their demand set along with the homogeneity of v_i (and thereby of u_i). If they preferred the other bundle (or a scaled version thereof), then they would want to buy that instead.

We can now easily prove that each allocation approach yields (budget-adjusted) envy freeness within each group. For unconstrained buyers the proof is the same for all the mechanisms: since unconstrained buyers all see the same price vector p, they already satisfy the affordability condition in Fact 1. We thus restrict each proof to handling the case of buyers that face constraints.

Proposition 1. EG with PBFP constraints yields per-group envy freeness.

Proof. Consider a pair of buyers i, i' that are constrained to satisfy PBFP. We know that their allocations $x_i, x_{i'}$ are such that $\sum_{j \in A} x_{ij} = \sum_{j \in B} x_{ij}$ and $\sum_{j \in A} x_{i'j} = \sum_{j \in B} x_{i'j}$. It follows that $\bar{p}_i^{\top} x_i = \bar{p}_{i'}^{\top} x_{i'} = 0$, and moreover $\bar{p}_i^{\top} x_{i'} = \sum_{j \in A} \lambda x_{i'j} - \sum_{j \in B} \lambda x_{i'j} = 0$, where λ is the Lagrange multiplier on the PBFP constraint of buyer i, which means that $\bar{p}_{ij} = \lambda$ for $j \in A$ and $\bar{p}_{ij} = -\lambda$ for $j \in B$. Since the only personalized price that i faces is the Lagrange multiplier on their PBFP constraint, it follows immediately that i can afford $(B_i/B_{i'})x_{i'}$ with leftover budget $B_i/B_{i'}\delta_{i'}$, and thus they have no budget-adjusted envy against i'.

Proposition 2. EG with PIFP constraints yields per-group envy freeness.

Proof. For any pair of buyers $i, i' \in A$, we have that they face the same price $p_j + \lambda_j$ for each item j, where λ_j is the Lagrange multiplier of the PIFP constraint. It immediately follows that i can afford $(B_i/B_{i'})x_{i'}$ with leftover budget $B_i/B_{i'}\delta_{i'}$. The same argument holds for pairs of buyers in group B.

Proposition 3. EG with AEF constraints yields per-group envy freeness.

Proof. For any pair of buyers $i, i' \in C$, we have that they face the same price $p_j + \lambda_j$ for each item j, where λ_j is the Lagrange multiplier of the floor constraint. It immediately follows that i can afford $(B_i/B_{i'})x_{i'}$ with leftover budget $B_i/B_{i'}\delta_{i'}$.

C Strategyproofness in the Large

First we survey the result of Azevedo and Budish (2018). This is a result for a general class of mechanisms that specify how agents can report types and how that leads to outcomes (which may include allocations as well as payments). We will consider a variant of their result covered in the appendix of their paper: we will need a more general outcome space, and semi-anonymous mechanisms. We use ΔT to denote the set of probability distributions over a given set T, and $\overline{\Delta T}$ to denote the set of distributions with full support. We note that this section overloads some notation that we used for Fisher markets; this is necessary in order to stay consistent with the notation of Azevedo and Budish (2018), and to aid readability in terms of sticking to conventional notation.

Next we specify the assumption made on the mechanism setting for the setting of Azevedo and Budish (2018). Let us call this the Azevedo-Budish setting. In the Azevedo-Budish setting, there is a finite set of types T and a measurable set of outcomes X_0 (see the appendix of Azevedo and Budish (2018) for the measurable outcomes case) that an individual agent may receive. For each type $t_i \in T$ there's a utility function $u_{t_i} : X \to [0, 1]$, where $X = \Delta X_0$ is the set of probability distributions over outcomes that an agent may receive. Note that in the Azevedo-Budish setup, any possible payments are included in X_0 and u_{t_i} , so we do not add an explicit extra variable corresponding to leftover budget.

The sets T and X_0 are held fixed for all market sizes. For each market size n, where n is the number of agents, there is a set $Y_n \subset (X_0)^n$ of feasible allocations. In our Fisher market setting Y_n is simply the set of supply-feasible allocations given the supply of items for a market of size n.

For a sequence of feasibility constraints $\{Y_n\}_n$ we say that a *direct mechanism* is a sequence of allocation functions $(\Phi^n: T^n \to \Delta((X_0)^n))_{n \in N}$ such that for all market sizes nand vectors of type reports $t \in T^n$, $\Phi^n(t)$ is contained in Y_n .⁵

We assume that the mechanism is *semi-anonymous*: the types space is partitioned into groups $T = \bigcup_{g \in G} T_g$. In our Fisher market setting, the groups will be the constrained and unconstrained buyers. Each agent of type $t \in T_g$ is restricted to reporting a type in T_g . For our constrained market setting, this corresponds to the fact that buyers cannot lie about whether they are constrained, but they may misreport their valuation function.

For each n we assume that we are given some type distribution $\mu \in \Delta T$ resulting e.g. from the underlying distribution over types composed with some, possibly randomized, map from types to report types. We then need the function $\phi^n : T \times \Delta T \to X$ which specifies the expected allocation function for each buyer given a reported type and a type distribution. This is defined to be

$$\phi^{n}(t_{i},\mu) = \sum_{t_{-i} \in T^{n-1}} \Phi^{n}_{i}(t_{i},t_{-i}) \cdot \Pr(t_{-i}|\mu).$$

Definition 1. A direct and semi-anonymous mechanism $\{\Phi^n\}_n$ is strategy-proof in the large (SPL) if, for any semianonymous type distribution with full support $\mu \in \overline{\Delta}T$ and $\epsilon > 0$, there exists n_0 such that for all $n \ge n_0$, all $g \in G$, and all $t_i, t'_i \in T_g$,

$$u_{t_i}(\phi^n(t_i,\mu)) \ge u_{t_i}(\phi^n(t'_i,\mu)) - \epsilon$$

i.e. there exists an n_0 such that for every type, the incentive to misreport is at most ϵ .

Azevedo and Budish (2018) show the following result for semi-anonymous direct mechanisms:

Theorem 1. Assume that we are in the Azevedo-Budish setting. If a mechanism is envy free within each group, then it is SPL. Given a type distribution with full support $\mu \in \overline{\Delta}T$ and $\epsilon > 0$, there exists C > 0 such that for all $g \in$ $G, t_i, t'_i \in T_g$ and n, the gain from deviating is bounded above by $C \cdot n^{-1/2+\epsilon}$.

Now we consider the following mechanism setting. Each agent reports a type (v_i, c) specifying a homogeneous, concave, and continuous valuation function v_i and a binary variable c denoting whether they are constrained or not (where they cannot lie about the constrained part). All agents are assumed to have budget $B_i = 1$, and all items are assumed to

have supply n/|T| (that is, supply grows linearly in the market size; the 1/|T| factor is WLOG. and for convenience). Then the mechanism computes an EG solution with an AEF constraint $\sum_{i \in C} \sum_{j \in P} x_{ij} \ge L(n, |\mathcal{C}|)$ (here we parameterize the floor L since we generally want the floor constraint to depend on the market size or the number of constrained buyers). Each agent then receives the corresponding allocation x_i and pays $1 - \delta_i$. If multiple agents reported the same type, then from the perspective of EG-AEF we treat them as a single representative buyer i with budget equal to the number of agents that reported the corresponding type, and split the allocation x_i proportionally. We call this the EG-AEF mechanism.

Assume that there is a finite set T of possible valuation functions v. Clearly the outcome space \mathbb{R}^m_+ is measurable, and the feasible set of allocation for market size n is the set of supply-feasible allocations and leftover budgets. Moreover, the mechanism is also obviously semi-anonymous: the EG-AEF program is invariant to permutation within the constrained and unconstrained buyer groups. Now, let $\mu \in \overline{\Delta}T$ be the full-support distribution of type reports, potentially resulting from the underlying distribution over types composed with the strategy used by each type. In terms of the Azevedo-Budish setup, we have that $X_0 = \mathbb{R}^{m+1}_+$ is the set of allocations vectors and leftover budget pairs (x_i, δ_i) for a given buyer i.

Azevedo and Budish (2018) assume that for each utility function u_{t_i} , $u_{t_i}(x) \in [0,1]$ for all $x \in X_0$. But this clearly cannot hold when $X_0 = \mathbb{R}^{m+1}_+$ and our valuation functions v are homogeneous, since if $v_i(x_i) > 0$ for any $(x_i, \delta_i) \in X_0$, then by homogeneity we can make $v_i(\alpha x_i)$ arbitrarily large by choosing a sufficiently-large α , thus violating the utility lying in [0, 1]. To alleviate this fact, consider a modified EG-AEF mechanism: we pick some large cutoff k such that $\mu_{t_i} > 1/k$, and then if the allocation under the reported type vector results in an allocation x such that for some agent i and item j we have $x_{ij} > k$ (after splitting proportionally among all agents that reported this type), then we throw away all the items and set $x_{ij} = 0, \delta = 0$ for all i, j. This modified EG-AEF mechanism is clearly still envy free: when we use an EG-AEF allocation we know that there is no envy within each group, and there's clearly no envy if no agent receives any allocation whatsoever. It follows that for any full-support type distribution μ , the modified EG-AEF mechanism satisfies the conditions of Theorem 1 and is thus SPL.

Now we can use this result to show that EG-AEF itself is SPL as a mechanism.

Theorem 2. The EG-AEF mechanism is SPL. Moreover, consider any distribution μ over types (v_i, c) with full support and an $\epsilon > 0$. Then there exists C > 0 such that for all constrained or unconstrained buyers, true valuation functions v, alternative valuation function reports v', and market sizes n, the gain from reporting v' rather than v is bounded above by $C \cdot n^{-1/2+\epsilon}$.

Proof. We already saw above that the modified EG-AEF mechanism is SPL. Now we show that the EG-AEF mechanism and the modified EG-AEF mechanism achieve the

⁵Azevedo and Budish (2018) also handle indirect mechanisms, but we do not need to consider that case here

same utility in an asymptotic sense.

Consider a type distribution μ with full support. Now consider a market size n and a sampled set of buyers $t = (v_1, c_1), \ldots, (v_n, c_n)$. If the sampled set of buyers are such that the resulting allocation satisfies $x_{ij} > k$ for some i, then the payoffs differ between EG-AEF and modified EG-AEF. Consider some particular type (v_i, c_i) in this scenario: under modified EG-AEF i gets utility zero. We can upper bound the utility that i gets under EG-AEF by the utility that they get from receiving *all* the items, i.e. $x_i = (n/|T|)\vec{1}$ and set $\delta_i = 1$, which is $u_i(x_i, \delta_i) = v_i(x_i) + 1 = (n/|T|)v_i(\vec{1}) + 1$ by homogeneity. Let $\bar{v} = \max_{(v,c)\in T} v(\vec{1})/|T|$, in which case we upper bound this by $n\bar{v} + 1$. So, in the worst case, a given type may gain $n\bar{v} + 1$ from misreporting in the case where we sample a set of types t such that EG-AEF and modified EG-AEF differ.

Let $x^*, \delta^*, x^{mod}, \delta^{mod}$ be the solutions from EG-AEF and modified EG-AEF, where we suppress the dependence on the reported types and market size.

Now fix an $\epsilon > 0$ and a type distribution with full support μ . It follows that for any type (v, c), they can gain at most

$$C \cdot n^{-1/2+\epsilon} + (n\bar{v}+1)\operatorname{Pr}(x^* \neq x^{mod}|\mu).$$

Thus, in order to show that EG-AEF is SPL, we need to show that $Pr(x^* \neq x^{mod}|\mu)$ gets small at a fast enough rate. In fact, we will show that this rate is exponential.

Suppose an agent with reported type t_i receives more than k units of some item j. In that case, we must have that in the corresponding EG-AEF allocation, the representative buyer i received k times the number of agents that reported type t_i of that good, let number of such reports be τ . First, this implies that the supply of the good is at least k, so this only happens when $n \ge k$. In the worst case, all n units of item j are allocated to the representative buyer i, in which case we need $n/\tau \ge k$. This implies that $\tau \le n/k$. Since the type reports of the other agents are distributed iid. according to μ , this means that we must sample at most n/k - 1 reports of type t_i for $x^* \ne x^{mod}$ to occur. The expected number of reports of type t_i is $n \cdot \mu_{t_i}$.

By Hoeffding's inequality, we now have that the probability of sampling n/k - 1 reports of type t_i or fewer is upper bounded by $2 \exp(-(n \cdot \mu_{t_i} - n/k + 1)^2/n)$. Since we chose k such that $\mu_{t_i} > 1/k$, this grows small exponentially fast in n.

The exact same proof goes through for PIFP and PBFP as well, because the result did not rely on any structure in the AEF constraint apart from the fact that for any pair of buyers i, i' from the same group, we have that they do not envy each other.

D Additional Examples

Please, see the tables on the next page for additional examples.

V	X^*	X*					X^{PIFP}			*	
BuyerItem C A 2 B .1	Item U 2 3	Buyer A B	Item C 1 0	Item <i>U</i> 0 1	$\begin{array}{c} B_i - px_i \\ 0 \\ 0 \end{array}$	$\begin{array}{c} u_i \\ 2 \\ 3 \end{array}$	Buyer A B	Item C .5 .5	Item U .258 .742	$\begin{array}{c} B_i - p x_i \\ 0 \\ 0 \end{array}$	u_i 1.52 2.28

Table 3: An example of a set of valuations with budgets set to 1 where imposing PIFP constraints on a subset of items leads to both originally disadvantaged (B) and originally advantaged (A) buyers being worse off.

V			X^*				X^{PBFP}				
Buyer C U	Item A 2 1	Item B 1 2	Buyer C U	Item A 1 0	Item B 0 1	$B_i - px_i $ 0 0	Buyer C U	Item A .5 .5	Item <i>B</i> .5 .5	$B_i - px_i $ 0 0	
U	1	2	U Price	0	1	0	U	.5	.5		

Table 4: An example of a set of valuations with budgets set to 1 where imposing PBFP constraints leads to both parity constrained (C) and unconstrained (U) buyers being worse off.

V			X^*				X^{PBFP}				
Buyer C U	Item A 2 0	Item B 2 2	Buyer C U	Item A 1 0	Item <i>B</i> 0 1	$\begin{array}{c} B_i - px_i \\ 0 \\ 0 \end{array}$	Buyer C U	Item A .5 .5	Item <i>B</i> .5 .5	$\begin{array}{c} B_i - px_i \\ 0 \\ 0 \end{array}$	

Table 5: An example of a set of valuations with budgets set to 1 where imposing PBFP constraints leads to an outcome that is Pareto suboptimal both in the buyer only case and the buyer-protected item case (here *B* is the originally disadvantaged group). X^{PBFP} is Pareto dominated in both cases by any allocation which transfers some item *A* to buyer *C*.

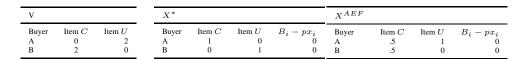


Table 6: An example of a set of valuations with budgets set to 1 where imposing AEF constraints of $x_{AC} \ge .5$ leads to buyer Pareto suboptimal outcomes.