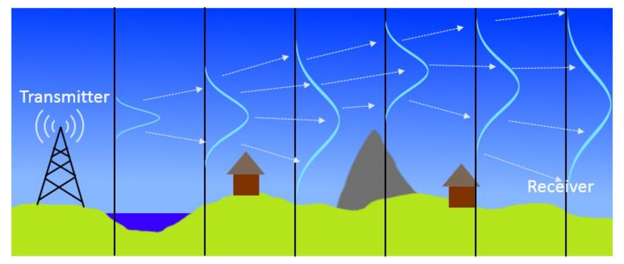
Gabor Frame-Based Sparsification and Radiation Boundary Conditions for Parabolic Wave Equations

Max Bright, Eric Michielssen, and Julius Kusuma
The University of Michigan, Ann Arbor, MI 48109
Facebook Connectivity Lab, Menlo Park, CA 94025

Introduction

- Parabolic Wave Equations are used extensively to model electromagnetic wave propagation over complex terrain and through the ionosphere
- Parabolic Equations are solved with the **Split-Step Fourier** method, which splits space into vertical slices [1]:



- FFT is used to advance through each slice

$$\psi(x+\Delta x, z) = IFFT\{\hat{\psi}(x,k_z)e^{ik_0\sqrt{1-\frac{k_z^2}{k_0}}\Delta x}\}e^{-i\Delta x(\mathbf{n}(z)k_0-k_0)}$$

- **Phase screens** are used to account for atmospheric effects

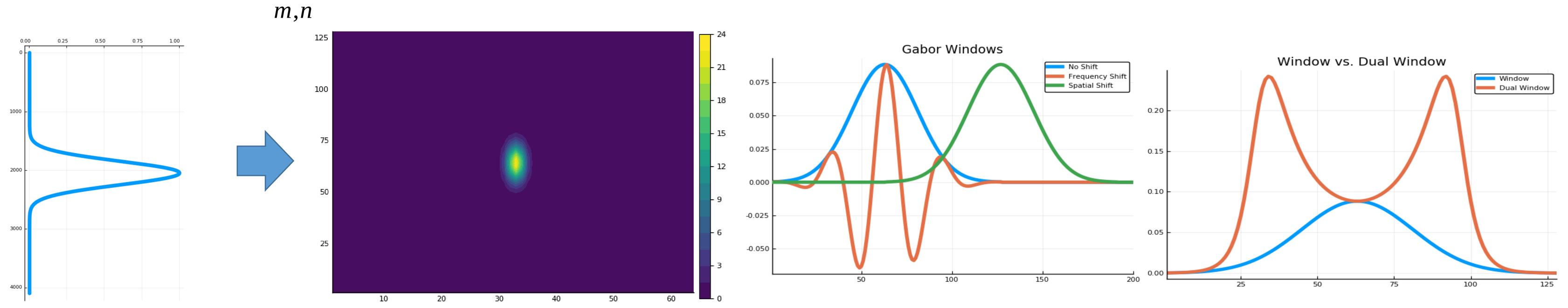
Pros/Cons of Split-Step Fourier

Pros	Cons
Handles wide range of angles	Memory scales with $O(N)$
Easy to implement	CPU time scales with $O(N \log N)$
	Entire domain must be stored
	RBCs are costly to implement

Gabor Transforms

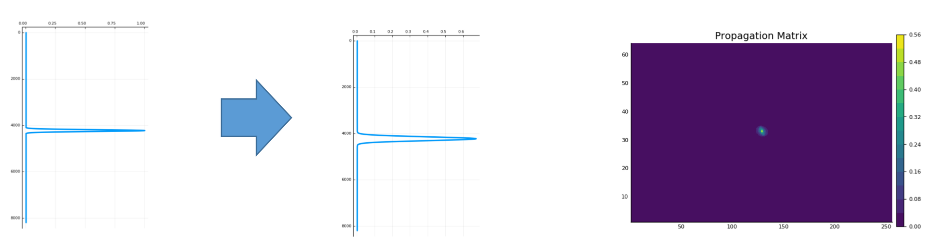
- Goal: A signal representation that promotes sparsity and easy RBC implementation
- Solution: Space-frequency technique to represent a wavefront
- The **Gabor Transform** decomposes wavefronts into weighted sums of window functions with **spatial shifts and frequency modulations** [2]
- Gabor coefficients are the inner product of wavefront and dual window

$$\psi(z) = \sum_{m,n}\mathbf{g}_{mn}(z)\langle\psi(z), \tilde{\mathbf{g}}_{mn}(z)\rangle \qquad \mathbf{g}_{mn}(z) = \mathbf{g}(z-na)e^{i(mb)z}$$

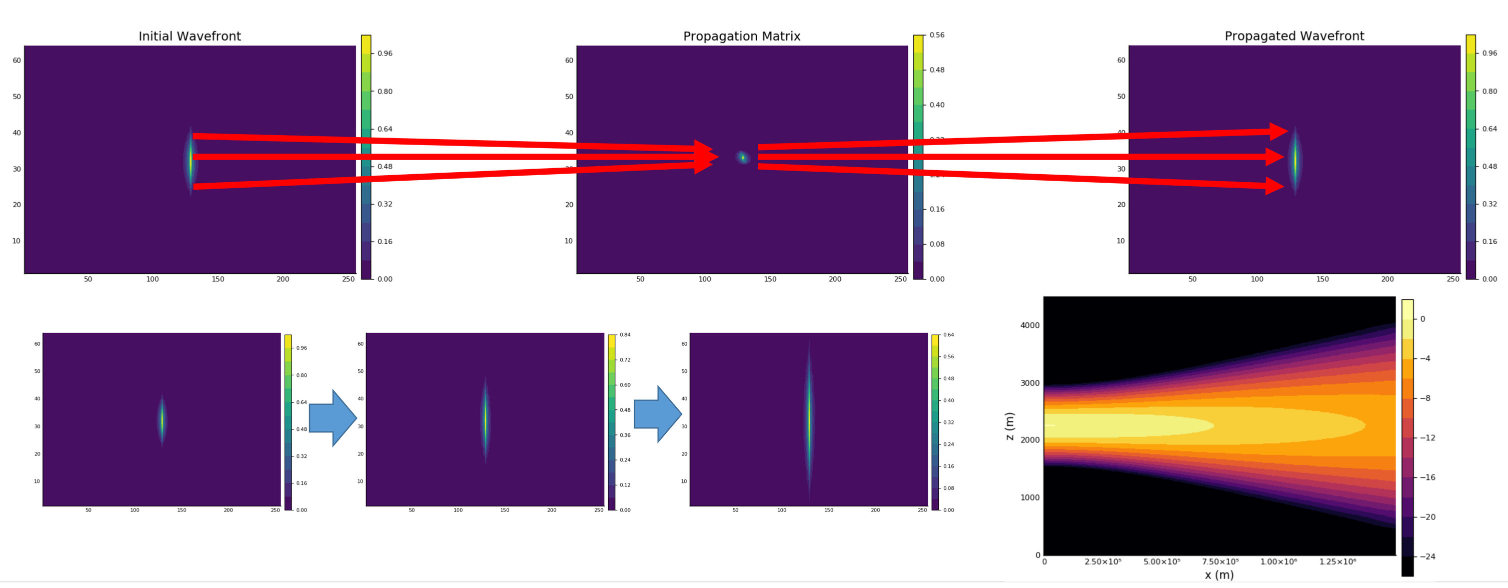


Propagation in the Gabor Domain

- Define a beamlet to be a window function propagating in free space.
- **Green's function (propagation matrix)** is computed by propagating each beamlet by one step [3]



- To advance a wavefront through space:
  - Take Gabor Transform of initial fields
  - Apply Green's function matrix to each Gabor coefficient
  - Inverse Gabor Transform to obtain propagated fields



Sparsification and Radiation Boundary Conditions (RBCs)

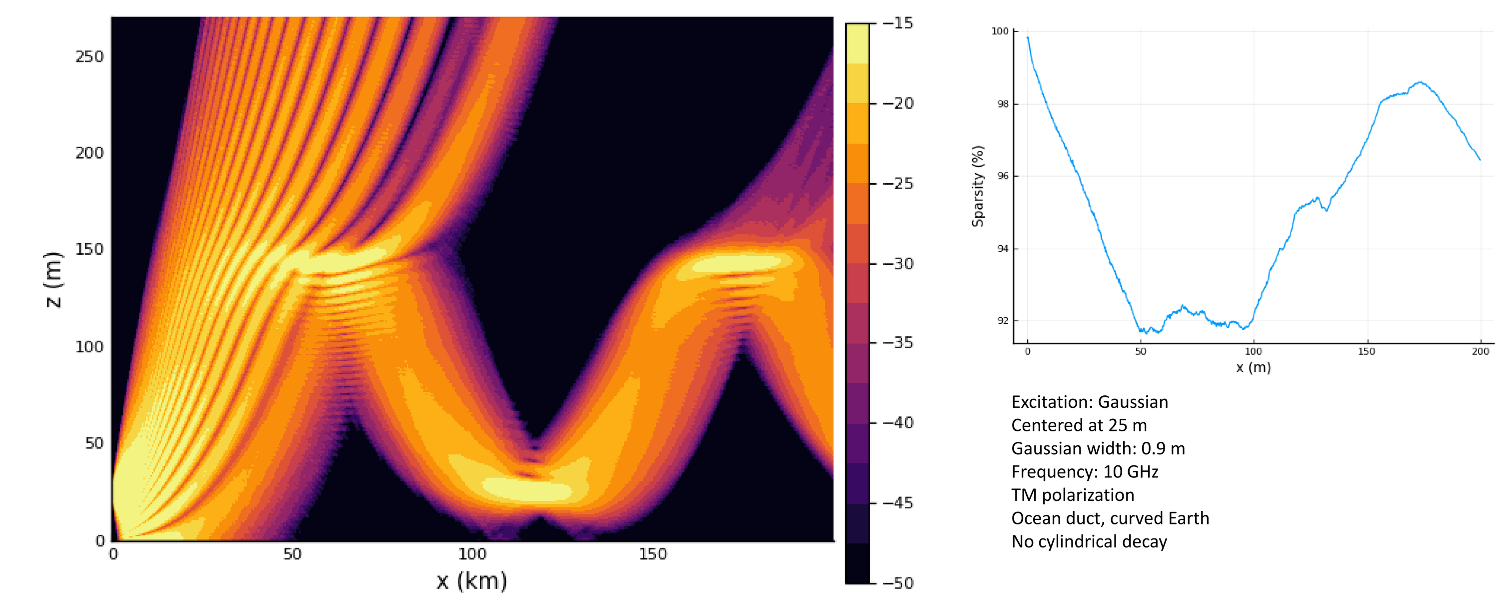
- The Gabor spectrum of a field can be sparsified via thresholding:

$$a_{mn}(x) = \begin{cases} a_{mn}(x), & |a_{mn}(x)| \geq \tau\|\mathbf{a}\|_2 \\ 0, & |a_{mn}(x)| < \tau\|\mathbf{a}\|_2 \end{cases}$$

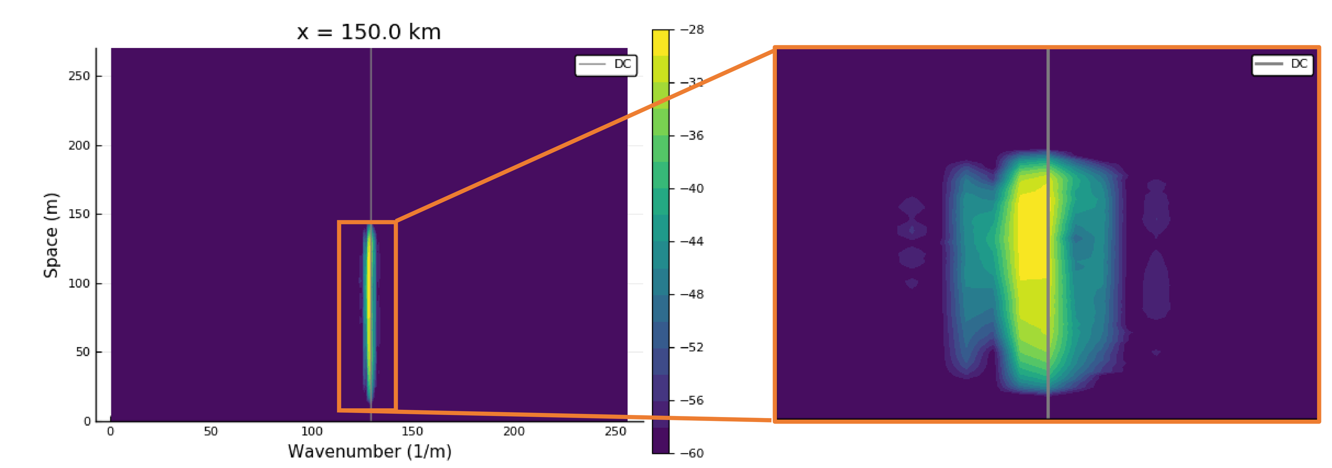
- CPU time and memory use per step is governed by the number of nonzero Gabor coefficients
- RBCs can be implemented by deleting window functions that "escape" the domain
- RBCs require no extra memory
- RBC implementation has minimal reflections at steep propagation angles
- RBC implementation has substantial reflections at grazing incidence to upper boundary
- Thin absorbing layer can be added to compensate:
  - Absorbing layers are effective for attenuating fields at grazing incidence
  - Absorbing layer is implemented by windowing fields at the edge of the domain

Propagation over Ocean

- Atmosphere modeled with trilinear duct
- Earth curvature correction applied
- Gabor field representation exceeds 90% sparsity with 11.4% average error

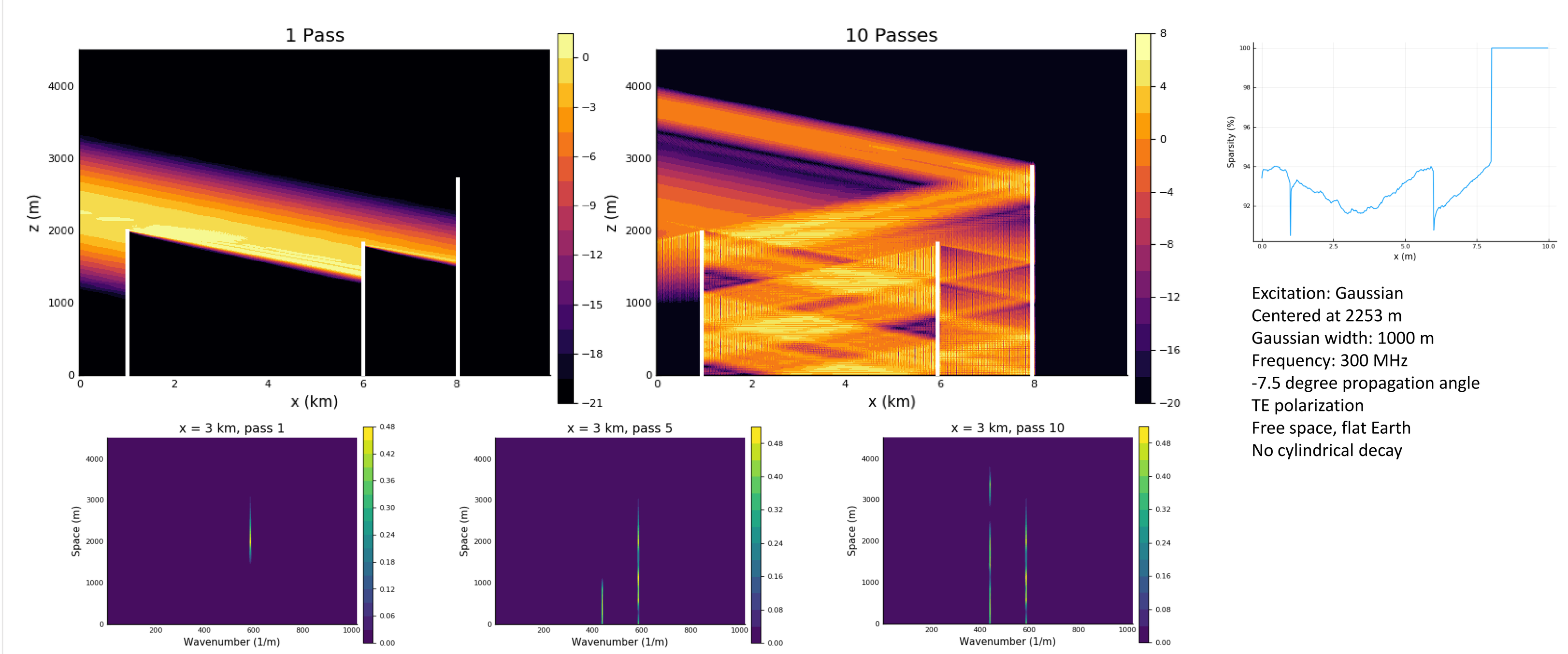


- Gabor coefficient spectrum shows modal behavior



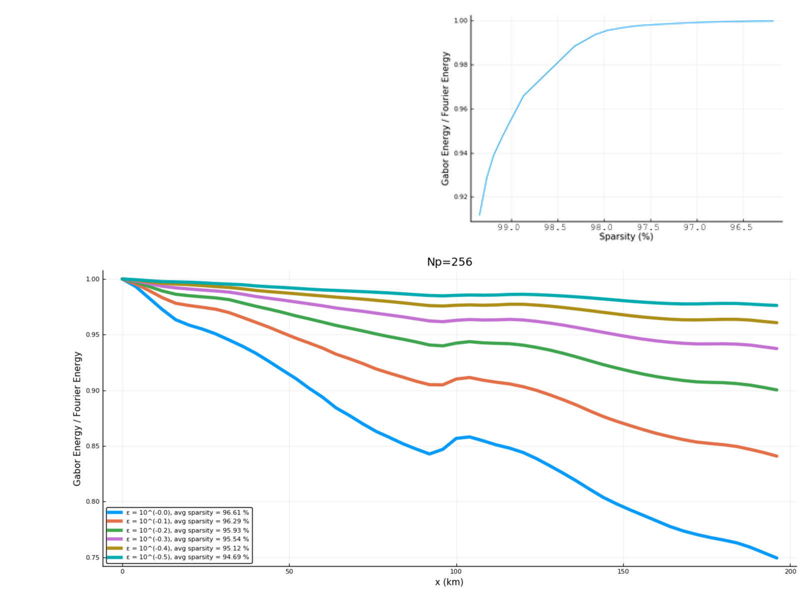
Knife-Edge Diffraction

- Modeled using Split-Step method enhanced with backward-forward propagation [4]:
  - Set fields on knife edges to zero, then reflect and propagate fields in opposite direction
- Gabor field representation exceeds 90% sparsity with 13.8% average error



Energy Loss from Sparsity

- Sparsifying fields results in a loss of energy
- The ratio between energy of the Split-Step Fourier method and the Gabor method is computed for the ocean ducting case:

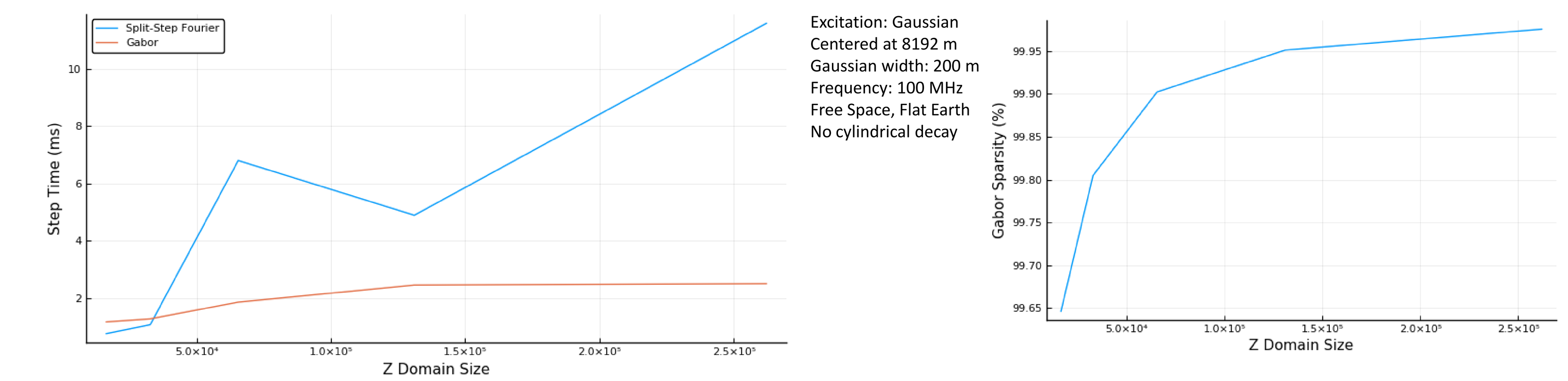


For a single spatial step:
Minimal loss for
up to 97-98% sparsity

For many spatial steps:
Energy loss accumulates
over long range, but this can
be controlled

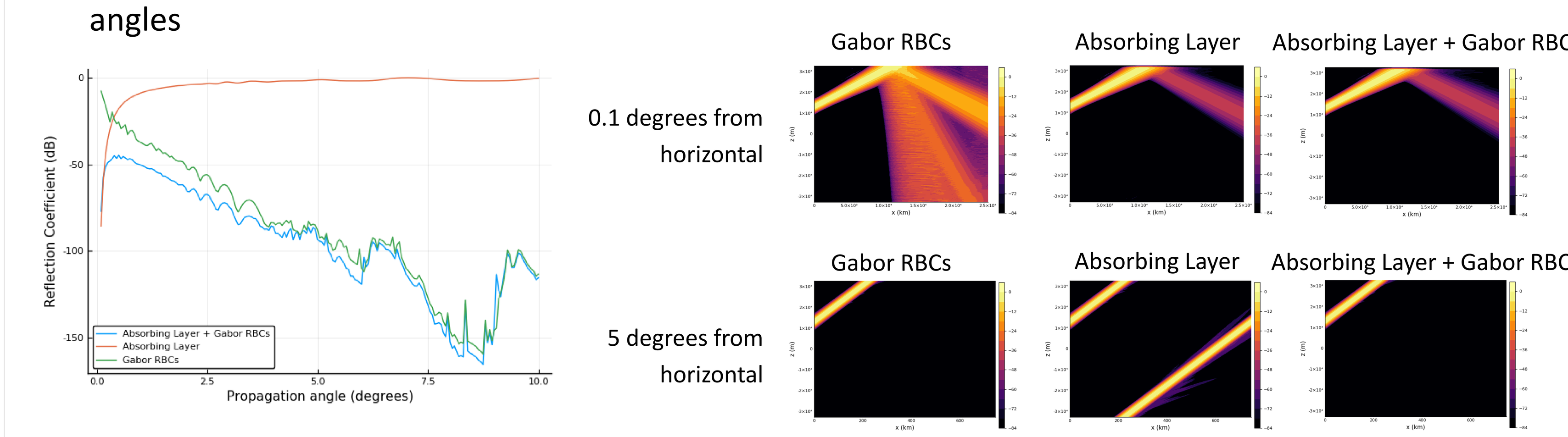
CPU Time: Gabor versus Split-Step Fourier

- CPU time for advancing a Gaussian beam by one spatial step is computed, where the Gabor parameters (e.g. window width, truncation threshold) are selected for maximum sparsity
- Gabor method outperforms Split-Step Fourier for large domains



Performance of Radiation Boundary Conditions

- Absorbing layer works well for paraxial propagation, but steep propagation angles require thick layer
- Gabor-based RBCs have minimal reflection for steep propagation angles
- Combined 2-km-thick absorbing layer + Gabor RBCs achieves minimal reflection for broad range of angles



- Reflection coefficient is computed by launching a beam at the upper boundary, letting energy reflect downwards or wrap around, and taking the ratio of final energy to initial energy

Conclusions

- Gabor propagator can be used as an alternative to Split-Step Fourier
- Gabor propagator is more efficient:
  - Structured fields have sparse representations
  - CPU and memory usage scales more efficiently than Fourier method
- Radiation Boundary Conditions are easily implemented:
  - Works well when paired with conventional absorbing layer
  - Efficient for all propagation angles

References

[1] M. Levy, Parabolic Equation Methods for Electromagnetic Wave Propagation. London: The Institution of Engineering and Technology, 2009.
[2] I. Daubechies, "The Wavelet Transform, Time-Frequency Localization and Signal Analysis," IEEE Trans. Inf. Theory, 1990.
[3] L. Chen, R. Wu, and Y. Chen, "Target-oriented beamlet migration based on Gabor-Daubechies frame decomposition," GEOPHYSICS, 2006.
[4] O. Ozgun, "Recursive two-way parabolic equation approach for modeling terrain effects in tropospheric propagation," IEEE Trans. Antennas Propag., 2009.